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Nuclear Physics B 882 (2014) 352–385

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Extremal surfaces and entanglement entropy

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Received 13 January 2014; received in revised form 20 February 2014; accepted 8 March 2014

Available online 15 March 2014

Editor: Hong-Jian He

Abstract

We have obtained the equation of the extremal hypersurface by considering the Jacobson–Myers functional and computed the entanglement entropy. In this context, we show that the higher derivative corrected extremal surfaces cannot penetrate the horizon. Also, we have studied the entanglement temperature and entanglement entropy for low excited states for such higher derivative theories when the entangling region is of the strip type.

1. Introduction

The study of the entanglement entropy in the AdS/CFT context [1] has attracted a lot of attention due to its potential application in condensed matter systems as well as in quantum information theories. In a seminal work, [2], Ryu and Takayanagi (RT) made a holographic conjecture of the computation of the entanglement entropy. The proposal is given for a static spacetime with a co-dimension two hypersurfaces, whose area is proposed to be related to the entanglement entropy. A proof of such a proposal is attempted in [3], with further comments in [4]. In another study, a proof is suggested in [5], when the entangling region is of the sphere type. More recently, a suggestive argument was put forward in [6] based on the previous works in $1+1$ dimensional CFT of [7] and [8] that explains the RT conjecture.

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http://dx.doi.org/10.1016/j.nuclphysb.2014.03.004

One of the universal result of [2] is that the entanglement entropy becomes divergent at UV, which goes as ϵ^{2-d}, with ϵ as the UV regulator, for a d dimensional CFT, and becomes finite at IR. In fact, the approach adopted in [2] is a non-covariant way of doing the calculation, which has been generalized in a covariant way in [9] by Hubeny, Rangamani and Takayanagi (HRT) to derive, in particular, the equation of the extremal hypersurface. To emphasize, this is the generalization of the minimal hypersurface in a covariant way of RT prescription. However, it is interesting to note that the equation of the hypersurface is also derived in [6] starting from the bulk Einstein's equation of motion, which is further studied in [10,11] and [12]. Some other interesting studies on entanglement entropy are reported in [13,14] and [15].

In this paper, we ask several questions and find answers to it. First, how would the equation of the hypersurface look, in a covariant way, upon inclusion of the higher derivative terms to the entanglement entropy functional? We answer this question, by considering the Jacobson–Myers functional (JM) [16] as the starting point for the holographic entanglement entropy functional and then derive the equation of the hypersurface by extremizing it with respect to the embedding fields,[1] X^S. It is important to note that such a from of the equation of the hypersurface does not depend on the shape or the size of the entangling region. As an example, with up to the Gauss–Bonnet term in the holographic entanglement entropy functional, the hypersurface reads as

$$\begin{aligned}&\mathcal{K}^S+\lambda_1\big(R\mathcal{K}^S-2R^{ab}\mathcal{K}^S_{ab}\big)+\Lambda\big[\mathcal{K}^S\big(R^2-4R_{a_1b_1}R^{a_1b_1}+R_{a_1b_1c_1d_1}R^{a_1b_1c_1d_1}\big)\\&\quad-4RR^{ab}\mathcal{K}^S_{ab}+8R^{acbd}R_{cd}\mathcal{K}^S_{ab}-4R^{aecd}R^{b}{}_{ecd}\mathcal{K}^S_{ab}+8R^{a}{}_{c}R^{bc}\mathcal{K}^S_{ab}\big]=0,\end{aligned}\tag{1}$$

where $\mathcal{K}^S\equiv g^{ab}\mathcal{K}^S_{ab}$, whose functional form is given in Eq. (38). The couplings λ_1 and Λ are undetermined constants and are dimensionfull objects. Let us recall, without turning on, Λ, the finite piece of the λ_1 dependent part of the entanglement entropy[2] is computed numerically in [17] for a 4 dimensional CFT. We revisit such a computation but for a d dimensional CFT, i.e., for bulk AdS spacetime and the result to the linear order in the coupling λ_1 can be summarized as follows: (a) The divergent term coming from UV goes in the same way as in the absence of the higher derivative term to the entanglement entropy functional, i.e., like ϵ^{2-d}, whereas (b) the finite term, the coefficient of $r_\star^{2-d}$, where $r_\star$ is the turning point, depends on the coupling λ_1 very non-linearly but we determine the functional form only to linear order. Explicitly, it reads as

$$\begin{aligned}2G_NS_{EE}&=\frac{L^{d-2}R_0^{d-1}}{\epsilon^{d-2}}\left(\frac{1-(d-1)(d-2)(\lambda_1/R_0^2)}{(d-2)}\right)\\&\quad-L^{d-2}R_0^{d-1}\frac{\sqrt{\pi}\,\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}r_\star^{2-d}\left(\frac{1+(d-2)(d-3)(\lambda_1/R_0^2)}{d-2}\right)+\mathcal{O}(\lambda_1)^2\\&=\frac{L^{d-2}R^{d-1}}{4\epsilon^{d-2}}\left(\frac{4-(d+1)(d-1)(d-2)a\lambda_a}{(d-2)}\right)\\&\quad-2^{d-2}L^{d-2}R^{d-1}\pi^{\frac{d-1}{2}}\left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\right)^{d-1}\\&\quad\times\ell^{2-d}\left(\frac{1+\frac{3}{4}(d-1)(d-2)(d-3)a\lambda_a}{d-2}\right)+\mathcal{O}(\lambda_a)^2,\end{aligned}\tag{2}$$

1 The embedding fields appear through the induced metric.

2 Unless stated otherwise, throughout the paper, we compute the entanglement entropy for strip type and for $d\geqslant 3$.

where R_0 and R are the radii with and without the higher derivative term to the AdS and $\lambda_1 = a\lambda_a R^2$ with $a = \frac{2}{(d-2)(d-3)}$. This value of a follows upon comparing our JM functional with Eq. (C.24) of [18] and[3] identifying the coupling $\lambda_a = \lambda_{there}$. After turning on both the couplings λ_1 and Λ, we find that the entanglement entropy for a $d = 6$ dimensional CFT follows the same pattern. That is apart from the UV divergent term, $1/\epsilon^4$, the finite term coming from IR goes $1/r_\star^4$ times some constants. Explicitly, the entanglement entropy to linear order in the couplings reads as

$$\begin{aligned} 2G_N S_{EE} &= \frac{L^4 R_0^5}{4\epsilon^4}\left(1 - \frac{20\lambda_1}{R_0^2} + \frac{120\Lambda}{R_0^4}\right) - \frac{4L^4 R_0^5}{11\ell^4}\pi^{5/2} \\ &\quad \times \left(11 + \frac{660\lambda_1}{R_0^2} - \frac{600\Lambda}{R_0^4}\right)\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{10})}\right)^5 \\ &= \frac{L^4 R^5}{4\epsilon^4}(1 - 35a\lambda_a + 300X_2\Lambda_a) \\ &\quad - \frac{4L^4 R^5}{11\ell^4}\pi^{5/2}(11 + 495a\lambda_a - 2460X_2\Lambda_a)\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{10})}\right)^5. \end{aligned} \tag{3}$$

In fact, we need to set $a = 1/6$ and $X_2 = -1/24$. This follows upon comparing with Eq. (C.24) of [18] and identifying the couplings as $\lambda_a = \lambda_{there}$ and $\Lambda_a = \mu_{there}$.

It simply follows from the expression of the entanglement entropy Eq. (A.10) or Eq. (A.13) that there exists a differential equation relating the entanglement entropy and the size, ℓ, as

$$\ell\frac{\partial}{\partial\ell}\left(\ell\frac{\partial}{\partial\ell} + (d-2)\right)S_{EE} = 0, \quad \text{with fixed } L \text{ and } R. \tag{4}$$

We ask the validity of such a differential equation upon inclusion of the higher derivative term to the holographic entanglement entropy functional? We check that such a form of the differential equation goes through to linear order in the coupling λ_1 and Λ. In fact, such a differential equation can be re-written as

$$\frac{\partial}{\partial\ell}\left(\ell^{d-1}\frac{\partial}{\partial\ell}S_{EE}\right) = 0. \tag{5}$$

In the context of RG flow, in [19], a candidate c-function has been suggested, which shows the necessary monotonicity along the flow from UV to IR as long as the matter fields obeys the null energy condition.[4] In fact, such a c-function[5] is proportional to $\ell^{d-1}\frac{\partial}{\partial\ell}S_{EE}$ for $d \geqslant 3$. The vanishing of the derivative of such a quantity with respect to the size, ℓ, suggests that we are sitting at the fixed point. However, in [21] another quantity is defined and called as "renormalized entanglement entropy" that gives the rate of flow.[6] Inspired by this, we find the simplest UV finite quantity is $\partial_\ell S_{EE}$. Using which, we find there exists another quantity, $\ell^2\partial_\ell^2 S_{EE} - \ell\partial_\ell S_{EE}$, which is negative but only when $d \geqslant 5$ for non-zero q in the perturbation to the geometry.

[3] The value of a makes sense for $d > 3$, but in the limit of $d \to 3$, the quantity $(d-3)a$ is finite.

[4] This holds good only when the bulk action is of the Einstein–Hilbert type but not for the Gauss–Bonnet type [19].

[5] For $d = 2$, the c-function is defined in [20].

[6] Such a quantity is defined strictly when the entangling region is of the sphere type, which is UV finite. In our case, with a slight abuse of notation, we define different quantities but use the same name and symbol as in [21].

In a recent study in [22] by Hubeny in the context of the entanglement entropy found that the spatial hypersurfaces does not penetrate the horizon. This motivate us to ask the following question: Is the non-penetrating of the horizon by the probe, spatial hypersurfaces, remain true even after the inclusion of the higher derivative term to the area functional?

The answer to this question is that upon inclusion of the higher derivative term of the type as in Eq. (36) to the action of the embedding field make the spatial hypersurface not to penetrate the horizon.

It is suggested recently, by studying the low excited states [23], that the entanglement entropy obeys a law like that of the first law of thermodynamics, $T_{ent}\Delta S = \Delta E$. The quantity T_{ent} is dubbed as the 'entanglement temperature', which is proportional to the inverse of the length, ℓ, i.e., $T_{ent} = c\ell^{-1}$. The proportionality constant, c, is a function of d, the spacetime dimensionality of the field theory and depends on the nature of the entangling region [23]. The question that we ask: Do we still expect to see a similar first law like relation even after the inclusion of the higher derivative term to the holographic entanglement entropy functional? If yes, then how does the 'entanglement temperature' go with the length, ℓ? And how does the proportionality constant behave as a function of d? We show that there indeed exists a first law like relation even with the higher derivative term to the holographic entanglement entropy functional and the relation between the 'entanglement temperature' and the size is the same as mentioned above. And the proportionality constant is a function of d and the coupling.[7]

$$\begin{aligned} c &= \frac{2(d^2-1)(\Gamma(\frac{d}{2(d-1)}))^2\Gamma(\frac{d+1}{2(d-1)})}{\sqrt{\pi}(\Gamma(\frac{1}{2(d-1)}))^2\Gamma(\frac{1}{(d-1)})}[1+4\lambda_a], \quad \text{for any } d \text{ with } \Lambda_a = 0, \\ c &= \frac{70\Gamma(\frac{7}{10})(\Gamma(\frac{3}{5}))^2}{187\sqrt{\pi}\Gamma(\frac{1}{5})(\Gamma(\frac{1}{10}))^2}(187+748\lambda_a+444\Lambda_a), \quad \text{for } d=6. \end{aligned} \tag{6}$$

The paper is organized as follows: In Section 2 and Appendix A, we re-visit the computation of the area functional of the hyperscaling violating geometries and generates both the existing [25–27] and new form of the geometry that exhibits the log violation of the entanglement entropy. In Section 3, we find the equation of the hypersurface using the technique of [9] but with the higher derivative term to the area functional and explicitly compute the entanglement entropy for AdS spacetime. In Section 4, we show following [22] the absence of the penetration of the horizon by the spatial hypersurfaces. In Section 5, we find the first law like of thermodynamics by considering fluctuation to the geometry and finally we conclude in Section 6. Some of the expression of the solution of the embedding field with the higher derivative term to the area functional has been relegated to Appendix B.

2. A differential equation

If we look at the expression of the holographic entanglement entropy, which is proportional to the area, Eq. (A.13) for AdS spacetime, then it follows that

$$\ell\frac{\partial}{\partial\ell}\left(\ell\frac{\partial}{\partial\ell}+(d-2)\right)S_{EE}=0. \tag{7}$$

[7] Similar type of question is asked in [24] for a 4 and 6 dimensional CFT when the entangling region is of the strip type and sphere type, respectively. Our study is different in the following way. We study the 6 dimensional CFT when the entangling region is of the strip type, which is not studied there. Also, we study the d dimensional CFT by considering the two derivative area functional whose special case, $d=4$, is studied in [24].

Note that in deriving such a differential equation we have considered a mild assumption that the UV cutoff, ϵ, is independent of the size ℓ, i.e., $\frac{d\epsilon}{d\ell}=0$. In fact, this is true because the size, ℓ depends only on $r_\star$. The details of the computation of S_{EE} are relegated to Appendix A. This type of differential equation would suggest some form of the 'RG flow' equation for the entanglement entropy. Strictly, such a form of the differential equation involving S_{EE} and ℓ follows when the entangling region is of the strip type and holds true irrespective of whether d is even or odd. In fact, a similar equation also follows for $r_\star$, as ℓ is linearly related to it. It reads as

$$r_\star\frac{\partial}{\partial r_\star}\left(r_\star\frac{\partial}{\partial r_\star}+(d-2)\right)S_{EE}=0. \tag{8}$$

It is expected that the limits of integration of the radial direction should be independent of each other. Hence, it is justified to consider that $r_\star$ and ϵ are not dependent on each other.

The simple looking differential equation (7) obeyed by S_{EE} for AdS spacetime gets changed when we change the background spacetime to Lifshitz type as follows

$$\ell\frac{\partial}{\partial \ell}\left(\ell\frac{\partial}{\partial \ell}+\big(d-2-\gamma(d-1)\big)\right)S_{EE}=0. \tag{9}$$

This follows from Eq. (A.21). From which it follows that

$$\frac{\partial}{\partial \ell}\left(\ell^{(d-1)(1-\gamma)}\frac{\partial S_{EE}}{\partial \ell}\right)=0. \tag{10}$$

In [21] a quantity, $\mathcal{S}_d^\Sigma$, is defined with the shape of Σ assumed to be a sphere. This has been used to study the rate of flow of the renormalized entanglement entropy. In our case of a strip type entangling region, we shall assume the existence of the following quantity: $\ell\frac{\partial \mathcal{S}_d^\Sigma}{\partial\ell}$, and check the consequence of imposition of $\ell\frac{\partial \mathcal{S}_d^\Sigma}{\partial\ell}<0$ on S_{EE} for $d=3$ and 4. To make it clear, we have taken the following forms, $\mathcal{S}_3^\Sigma=\ell\partial_\ell S_{EE}-S_{EE}$ and $\mathcal{S}_4^\Sigma=\frac{\ell}{2}\partial_\ell(\ell\partial_\ell S_{EE}-2S_{EE})$. Note that $\mathcal{S}_3^\Sigma$ is not UV finite unlike $\mathcal{S}_4^\Sigma$.

On imposition of $\ell\frac{\partial \mathcal{S}_d^\Sigma}{\partial\ell}<0$ for $d=3,4$, we find

$$\ell\partial_l S_{EE}>0 \quad \text{for } d=3, \quad \text{and} \quad \ell\partial_l S_{EE}<0 \quad \text{for } d=4. \tag{11}$$

In getting such a result, we have used Eq. (7). Upon evaluating $\ell\partial_l S_{EE}$ explicitly, using Eq. (A.13), we find it to be positive for any d. For $d=3$, this result is in some sense consistent with [21] because the authors did not find any example that violates the monotonicity of $\mathcal{S}_3$ unlike $\mathcal{S}_4$.

In what follows, we shall define two more quantities, which are UV finite, $\ell\partial_\ell(\ell\partial_\ell S_{EE})$ and $\ell^3\partial_\ell(\ell^{-1}\partial_\ell S_{EE})$ and find whether they can be used to study the rate of flow, i.e., are negative? Given the explicit result of the entanglement entropy for AdS spacetime in Eq. (A.13), we find the following equations for any $d\geqslant 3$

$$\begin{aligned}&\ell\partial_\ell(\ell\partial_\ell S_{EE})=\ell^2\partial_\ell^2 S_{EE}+\ell\partial_\ell S_{EE}<0,\\&\ell^3\partial_\ell\big(\ell^{-1}\partial_\ell S_{EE}\big)=\ell^2\partial_\ell^2 S_{EE}-\ell\partial_\ell S_{EE}<0.\end{aligned} \tag{12}$$

In fact there exists the following relation

$$\ell\partial_\ell\mathcal{S}_3^\Sigma=\frac{\ell\partial_\ell(\ell\partial_\ell S_{EE})+\ell^3\partial_\ell(\ell^{-1}\partial_\ell S_{EE})}{2}. \tag{13}$$

Hence, it is not surprising to see that $\ell\frac{\partial\mathcal{S}_3^\Sigma}{\partial\ell}<0$. Hence, it is highly plausible to consider these two quantities, which may give information on the rate of flow. This is further investigated by doing fluctuations to the geometry in Subsection 5.3 and there arises interesting restrictions on d.

2.1. Log term to S_{EE}

In this section, we shall try to find answer to the following question: Can we generate log term in the entanglement entropy? Let us assume the following form of the metric in $3+1$ dimensional spacetime

$$ds_{3+1}^2=-g_{tt}(r)\,dt^2+g_{xx}(r)\,dx^2+g_{yy}(r)\,dy^2+g_{rr}(r)\,dr^2+2g_{xy}(r)\,dx\,dy, \tag{14}$$

we assume that the boundary is at $r=0$. Following the proposal of [2], the geometry of the co-dimension two hypersurface becomes

$$ds_2^2=\big(g_{xx}+g_{rr}r'^2(x)\big)\,dx^2+g_{yy}\,dy^2+2g_{xy}\,dx\,dy \tag{15}$$

So the area of the hypersurface becomes

$$\mathcal{A}=2\int dy\,dr\,\sqrt{\big(g_{xx}g_{yy}-g_{xy}^2\big)x'^2(r)+g_{rr}g_{yy}}, \tag{16}$$

where we have inverted the function $r(x)$ and written as $x(r)$. Now the hypersurface $x(r)$ that extremizes the area $\mathcal{A}$ is

$$\frac{dx}{dr}=\frac{\sqrt{(g_{xx}(r_\star)g_{yy}(r_\star)-g_{xy}^2(r_\star))}\sqrt{g_{yy}(r)g_{rr}(r)}}{\sqrt{[g_{xx}(r)g_{yy}(r)-g_{xy}^2(r)][g_{xx}(r)g_{yy}(r)-g_{xy}^2(r)-g_{xx}(r_\star)g_{yy}(r_\star)+g_{xy}^2(r_\star)]}}, \tag{17}$$

where the turning point $r_\star$ is determined when the quantity $x'(r_\star)$ diverges. We consider the entangling region to be like a strip, $0\leqslant x\leqslant\ell$ and $-L/2\leqslant y\leqslant L/2$ and the size

$$\frac{\ell}{2}=\int_0^{r_\star}dr\,\frac{\sqrt{(g_{xx}(r_\star)g_{yy}(r_\star)-g_{xy}^2(r_\star))}\sqrt{g_{yy}(r)g_{rr}(r)}}{\sqrt{[g_{xx}(r)g_{yy}(r)-g_{xy}^2(r)][g_{xx}(r)g_{yy}(r)-g_{xy}^2(r)-g_{xx}(r_\star)g_{yy}(r_\star)+g_{xy}^2(r_\star)]}}. \tag{18}$$

In such a case, the extremized area turns out to be

$$\mathcal{A}=2L\int_\epsilon^{r_\star}dr\,\frac{\sqrt{g_{rr}(r)g_{yy}(r)}}{\sqrt{1-\frac{g_{xx}(r_\star)g_{yy}(r_\star)-g_{xy}^2(r_\star)}{g_{xx}(r)g_{yy}(r)-g_{xy}^2(r)}}}, \tag{19}$$

where ϵ is the UV-cutoff. From now on wards, we shall be working in the diagonal form of the bulk metric i.e., $g_{xy}=0$, for simplicity. From this expression of the area, we can ask: under what condition do we see a log term? The condition to see such a log term, using Eq. (A.22) are

$$g_{rr}g_{yy}=\frac{1}{r^{2(2k+1)}},\qquad g_{xx}g_{yy}=\frac{1}{r^2},\quad\text{for } k=0,1,2,\ldots. \tag{20}$$

In which case the area becomes

$$\mathcal{A}(k) = 2LR^2 \int_{\epsilon}^{r_\star} \frac{dr}{r^{2k+1}} \frac{1}{\sqrt{1-(r/r_\star)^2}}, \quad \text{for } k = 0, 1, 2, \ldots. \tag{21}$$

We have put an index k in the area to label it. In order to fix the metric components, let us assume that

$$g_{yy} = r^{-2w}, \qquad g_{xx} = r^{2(w-1)}, \qquad g_{rr} = r^{2(w-2k-1)}, \tag{22}$$

in which case the $3+1$ dimensional geometry reads as

$$ds^2_{3+1}(k, w) = R^2\left[-g_{tt}(r)\,dt^2 + \frac{dx^2}{r^{2(1-w)}} + \frac{dy^2}{r^{2w}} + \frac{dr^2}{r^{2(2k+1-w)}}\right], \tag{23}$$

where we have used the indices k and w to label the geometry. Moreover, the entanglement entropy does not fixes the time–time component of the metric tensor. The quantity

$$\ell/2 = \frac{1}{r_\star} \int_0^{r_\star} \frac{dr}{r^{2k-1}} \frac{1}{\sqrt{1-(r/r_\star)^2}}. \tag{24}$$

Note, there exists UV divergence to ℓ for $k \geqslant 1$. Once again to regulate it, we have put an UV cutoff. In which case,

$$\frac{\ell}{2} = \frac{1}{r_\star} \int_{\epsilon}^{r_\star} \frac{dr}{r^{2k-1}} \frac{1}{\sqrt{1-(r/r_\star)^2}}, \quad \text{for } k \geqslant 1. \tag{25}$$

Explicitly, the area for $k=0$ and $k=1$ are

$$\begin{aligned} &\mathcal{A}(k=0) = 2LR^2 \,\mathrm{Log}\left(\frac{2r_\star}{\epsilon}\right), \qquad \frac{\ell}{2} = r_\star, \\ &\mathcal{A}(k=1) = 2LR^2\left[\frac{1}{2r_\star^2}\mathrm{Log}\left(\frac{2r_\star}{\epsilon}\right) - \frac{1}{4r_\star^2} + \frac{1}{2\epsilon^2}\right], \qquad \frac{\ell}{2} = \frac{1}{r_\star}\mathrm{Log}\left(\frac{2r_\star}{\epsilon}\right) \end{aligned} \tag{26}$$

Note for $k=0$ case, the entanglement entropy obeys Eq. (10) for $\gamma = 1/2$.

It is interesting to note that the neither the area Eq. (21) nor the length, ℓ, Eq. (24) depends on w. It means there can be more than one co-dimension two geometry which has the same area and the length ℓ. The explicit from of the bulk geometry for $k = 0, 1$ are

$$\begin{aligned} ds^2_{3+1}(k=0, w) &= R^2\left[-g_{tt}(r)\,dt^2 + \frac{dx^2}{r^{2(1-w)}} + \frac{dy^2}{r^{2w}} + \frac{dr^2}{r^{2(1-w)}}\right], \\ ds^2_{3+1}(k=1, w) &= R^2\left[-g_{tt}(r)\,dt^2 + \frac{dx^2}{r^{2(1-w)}} + \frac{dy^2}{r^{2w}} + \frac{dr^2}{r^{2(5-w)}}\right] \end{aligned} \tag{27}$$

For $k=1$ case, it is very difficult to find the explicit dependence of the area in terms of ℓ and the UV cutoff ϵ. Even though the area depends logarithmically on $r_\star$, but it is not clear it will do so in terms of ℓ. Also, equation of the type, Eq. (10), is difficult to satisfy for $\gamma = 3/2$, where γ is defined below in Eq. (30).

Looking at the geometries for $k=0$ and 1, we see the presence of rotational symmetry only for $w = 1/2$. Hence, let us find out the scaling behavior of such cases.

Case 1. $w = 1/2$. For this choice of w, there exists a rotational symmetry, i.e., $g_{xx} = g_{yy}$. The choice $k = 0$ corresponds to that studied in the previous section and also found in [26]. The other choices of $k \geqslant 1$ are new.

As an example, let us looks at the explicit geometry for $k = 1$. In which case, we get

$$g_{xx} = g_{yy} = \frac{R^2}{r}, \qquad g_{rr} = \frac{R^2}{r^5}. \tag{28}$$

It means, we can write down the geometry as

$$\begin{aligned} ds_4^2 &= R^2\left[-g_{tt}(r)\,dt^2 + \frac{(dx^2+dy^2)}{r} + \frac{dr^2}{r^{4k+1}}\right] \\ &= R^2 r\left[-\tilde{g}_{tt}(r)\,dt^2 + \frac{(dx^2+dy^2)}{r^2} + \frac{dr^2}{r^{2(2k+1)}}\right], \end{aligned} \tag{29}$$

which falls under the category of the hyperscaling violating geometry provided t has a nice scaling behavior. It is easy to see that for $k = 0$ the geometry is the same as written in Eq. (A.14) for $d = 3$ and $\gamma = 1/2$. If we assume that $g_{tt} = \rho^{\frac{1-4k+2z(1-2k)}{2k-1}}$, then for generic k we can have the following scaling behavior

$$\rho \to \frac{\rho}{\lambda}, \qquad t \to \lambda^{-z} t, \qquad x^i \to \lambda x^i, \qquad ds \to \lambda^{\frac{4k-1}{2(2k-1)}}\,ds \equiv \lambda^{\gamma}\,ds, \tag{30}$$

where $\rho = r^{2k-1}$.

Case 2. This corresponds to those solutions which does not respect the rotational symmetry i.e., solutions other than $w = 1/2$. Let us take different choices of w. In which case, the $3+1$ dimensional solution becomes

$$\begin{aligned} ds_{3+1}^2 &= R^2\left[-g_{tt}(r)\,dt^2 + \frac{dx^2}{r^2} + dy^2 + \frac{dr^2}{r^{2(2k+1)}}\right], \quad \text{for } w = 0 \\ &= R^2\left[-g_{tt}(r)\,dt^2 + \frac{dy^2}{r^2} + dx^2 + \frac{dr^2}{r^{4k}}\right], \quad \text{for } w = 1. \end{aligned} \tag{31}$$

For $g_{tt} = r^{-2}$ with $k = 0$ and $w = 0$, it corresponds to $AdS_3 \times R^1$ or $AdS_3 \times S^1$ for non-compact and compact, y, respectively. As we saw earlier, the entanglement entropy does not depend on w and for $k = 0$ case it goes as $S_{EE} \sim \mathrm{Log}(\ell/\epsilon)$. It means for $k = 0$ and $w = 0$, we can have a log violation to the entanglement entropy as well.

Subsummary. We obtain the presence of the logarithmic term in the entanglement entropy in $3+1$ dimensional bulk system with the rotational symmetry along the spatial directions. In fact the only geometry that is found here, corresponding to having $k = 0$, and is the same as that found in [26], which is of the Lifshitz type. So, the only rotationally invariant solution whose entanglement entropy goes logarithmically with the size, ℓ, corresponds to the $k = 0$ case.

In the absence of the rotational symmetry, we have obtained several geometries as written in Eq. (31). In particular, the direct product of geometries like $AdS_3 \times R^1$ or $AdS_3 \times S^1$ shows the presence of log term in S_{EE}.

2.2. Re-visit of S_{EE} as studied in [17]

In [17], the authors considered a geometry that does not have the full rotational symmetry $SO(d-1)$ in a $d+1$ dimensional bulk system, while studying the entanglement entropy. We are going to re-investigate this calculation but without the higher derivative correction.

Let us substitute in Eq. (19) the following form of the metric components as considered in [17]

$$g_{tt} = g_{xx} = g_{rr} = 1/r^2, \qquad g_{yy} = 1/r^{2w}, \qquad g_{xy} = 0. \tag{32}$$

Now the hypersurface is described by $x(r)$, whose explicit form can be read out from Eq. (17). On computing the area of the hypersurface of a strip type entangling region

$$\begin{aligned}\frac{\mathcal{A}}{2} &= L\int_{\epsilon}^{r_\star} dr\,\frac{1}{r^{w+1}\sqrt{1-(\frac{r}{r_\star})^{2(w+1)}}}\\ &= -\left(\frac{L}{wr^w}{}_2F_1\left[\frac{1}{2}, -\frac{w}{2(w+1)}, \frac{w+2}{2(w+1)}, \left(\frac{r}{r_\star}\right)^{2(w+1)}\right]\right)_{\epsilon}^{r_\star}\\ &= \frac{L}{w\epsilon^w} - \frac{L\sqrt{\pi}}{(2w+1)r_\star^w}\frac{\Gamma(-\frac{w}{2(w+1)})}{\Gamma(-\frac{2w+1}{2(w+1)})},\end{aligned} \tag{33}$$

and the size

$$\frac{\ell}{2} = r_\star\sqrt{\pi}\frac{\Gamma(\frac{2+w}{2(1+w)})}{\Gamma(\frac{1}{2(1+w)})}. \tag{34}$$

Note that the $3+1$ dimensional geometry as written above is a solution at IR not at UV. Hence, it is expected that such a solution will not show the desired result, ϵ^{-1}, at UV. In such a case, the entanglement entropy obeys the following differential equation

$$\ell\frac{\partial}{\partial\ell}\left(\ell\frac{\partial}{\partial\ell} + w\right)S_{EE} = 0. \tag{35}$$

3. The equation of the hypersurface

In this section, we shall derive the covariant equation of the extremal hypersurface with higher derivative effects. It means, we are including the effects of the finite ’t Hooft coupling. The equation of motion of the embedding fields, $X^M(\sigma^a)$ essentially gives the form of the hypersurface. The induced metric is given by $g_{ab} = \frac{\partial X^M}{\partial\sigma^a}\frac{\partial X^N}{\partial\sigma^b}G_{MN}$, where G_{MN} denotes the $d+1$ dimensional geometry of the bulk spacetime, σ^a are the coordinates on the codimension-2 hypersurface. Let us assume that the entanglement entropy functional is

$$\begin{aligned}4G_N S_{EE} = \int d^{d-1}\sigma\sqrt{\det(g_{ab})}\big[&1+\lambda_1 R(g)+\lambda_2 R^2(g)+\lambda_3 R_{ab}(g)R^{ab}(g)\\ &+\lambda_4 R_{abcd}(g)R^{abcd}(g)\big]\end{aligned} \tag{36}$$

where we have included higher derivative correction with λ_i’s as the coefficients.[8]

[8] In [36], another kind of higher derivative term in the entanglement entropy functional is studied.

A priori there is no good reason to believe the inclusion of higher derivative terms in this particular way. Even though this is purely a guess but we hope, it can be thought of as follows. For the Einstein–Hilbert action, it is suggested in [2] to consider only the first term in Eq. (36), i.e., setting all the λ_i's to zero. Upon inclusion of the Gauss–Bonnet term to the (bulk) Einstein–Hilbert action, it is suggested in [17] that the entanglement entropy should be given by Eq. (36) for which $\lambda_2 = 0 = \lambda_3 = \lambda_4$. Hence, it follows from these examples that for each power of the 'Ricci scalar' in the bulk theory[9] one should include a 'Ricci scalar' with one less power in the entanglement entropy. Note that the 'Ricci scalar' in the entanglement entropy should be constructed out of the induced metric g_{ab}. Formally, we can write it as

$$\begin{aligned}
&\int \sqrt{G}\left[\mathcal{R}(G) + \frac{d(d-1)}{R^2}\right] \longrightarrow \int \sqrt{g},\\
&\int \sqrt{G}\left[\mathcal{R}(G) + \lambda \mathcal{GB}(G)\right] \longrightarrow \int \sqrt{g}\left[1 + 2\lambda R(g)\right],
\end{aligned} \tag{37}$$

where the left hand side is the bulk action and right hand side is the entanglement entropy functional and $\mathcal{GB}(G)$ denotes the Gauss–Bonnet term made from the bulk metric G_{MN}. Similarly, if we go for one more higher power of the scalar curvature then it is highly plausible to consider the terms as written in Eq. (36) with arbitrary coefficients. In [18], the authors have considered the Jacobson–Myers form of the entropy functional [16] and studied the entanglement entropy when the entangling region is of the sphere and cylinder type. In our study, we do it for the strip type. Before moving onto the calculation of the extremal surface, recently in [14], the author has given a derivation of the Jacobson–Myers entropy functional.

On varying the entanglement entropy functional, Eq. (36), with respect to the embedding field, X^S, gives

$$\begin{aligned}
&X^{ab}\mathcal{K}^S_{ab} + \partial_b X^S \nabla_a X^{ab} = 0,\\
&\quad \text{where } \mathcal{K}^S_{ab} \equiv \partial_a\partial_b X^S - \gamma^c_{ab}\partial_c X^S + \partial_a X^M \partial_b X^N \Gamma^S_{MN}
\end{aligned} \tag{38}$$

where the γ^c_{ab} and Γ^K_{MN} are connections defined with respect to g_{ab} and G_{MN}, respectively. In fact, $\mathcal{K}^S_{ab}$ obeys an identity $\mathcal{K}^M_{ab}\partial_c X^N G_{MN} = 0$. The quantity

$$\begin{aligned}
X^{ab} = {}& \frac{1}{2}g^{ab} + \lambda_1\left(\frac{1}{2}g^{ab}R - R^{ab}\right)\\
& + \lambda_2\left(\frac{1}{2}g^{ab}R^2 - 2RR^{ab} + \nabla^a\nabla^b R + \nabla^b\nabla^a R - 2g^{ab}\nabla^2 R\right)\\
& + \lambda_3\left(\frac{1}{2}g^{ab}R_{cd}R^{cd} + \frac{1}{2}\nabla^a\nabla^b R + \frac{1}{2}\nabla^b\nabla^a R\right.\\
& \left. - \frac{1}{2}g^{ab}\nabla^2 R - 2R^{acbd}R_{cd} - \nabla^2 R^{ab}\right)\\
& + \lambda_4\left(\frac{1}{2}g^{ab}R_{a_1b_1c_1d_1}R^{a_1b_1c_1d_1} - 2R^{aecd}{R^b}_{ecd} - 4\nabla^2 R^{ab}\right.\\
& \left. + \nabla^a\nabla^b R + \nabla^b\nabla^a R - 4R^{acbd}R_{cd} + 4R^a_c R^{bc}\right).
\end{aligned} \tag{39}$$

[9] Here we mean by 'Ricci scalar' are the terms with all possible combination of Riemann tensor, Ricci tensor as well as Ricci scalar that is diffeo invariant.

For this form of X^{ab}, one can easily show that, it obeys an identity: $\nabla_a X^{ab}=0$, where the covariant derivative is defined with respect to g_{ab}. In which case, the equation of motion of the extremal hypersurface becomes

$$X^{ab}\mathcal{K}^S_{ab}=0. \tag{40}$$

Note, the equation of the hypersurface is independent of the shape and size of the entangling region. Now, we shall write down the form of X^{ab} in two different cases.

Gauss–Bonnet combination. Let us consider a very specific combination where $\lambda_2=\Lambda$, $\lambda_3=-4\Lambda$, $\lambda_4=\Lambda$, then X^{ab} takes the following form

$$\begin{aligned} X^{ab}={}&\frac{1}{2}g^{ab}+\lambda_1\left(\frac{1}{2}g^{ab}R-R^{ab}\right)+\Lambda\left[\frac{1}{2}g^{ab}\left(R^2-4R_{a_1b_1}R^{a_1b_1}+R_{a_1b_1c_1d_1}R^{a_1b_1c_1d_1}\right)\right.\\ &\left.-2RR^{ab}+4R^{acbd}R_{cd}-2R^{aecd}R^b{}_{ecd}+4R^a_cR^{bc}\right]. \end{aligned} \tag{41}$$

In which case, the equation of motion of X^S can be re-written as

$$\begin{aligned} &\mathcal{K}^S+\lambda_1\left(R\mathcal{K}^S-2R^{ab}\mathcal{K}^S_{ab}\right)+\Lambda\left[\mathcal{K}^S\left(R^2-4R_{a_1b_1}R^{a_1b_1}+R_{a_1b_1c_1d_1}R^{a_1b_1c_1d_1}\right)\right.\\ &\quad\left.-4RR^{ab}\mathcal{K}^S_{ab}+8R^{acbd}R_{cd}\mathcal{K}^S_{ab}-4R^{aecd}R^b{}_{ecd}\mathcal{K}^S_{ab}+8R^a{}_cR^{bc}\mathcal{K}^S_{ab}\right]=0, \end{aligned} \tag{42}$$

where $\mathcal{K}^S\equiv g^{ab}\mathcal{K}^S_{ab}$.

Weyl-square combination. In this case, the λ_i's take the following values: $\lambda_2=\frac{2\Lambda}{(d-2)(d-3)}$, $\lambda_3=-\frac{4\Lambda}{d-3}$ and $\lambda_4=\Lambda$. In which case the X^{ab} takes the following form

$$\begin{aligned} X^{ab}={}&\frac{1}{2}g^{ab}+\lambda_1\left(\frac{1}{2}g^{ab}R-R^{ab}\right)+\Lambda\left[\frac{1}{2}g^{ab}\left(\frac{2}{(d-2)(d-3)}R^2-\frac{4}{d-3}R_{a_1b_1}R^{a_1b_1}\right.\right.\\ &\left.+R_{a_1b_1c_1d_1}R^{a_1b_1c_1d_1}\right)-\frac{4}{(d-2)(d-3)}RR^{ab}\\ &+\frac{d-4}{d-2}\nabla^a\nabla^bR+\frac{d-4}{d-2}\nabla^b\nabla^aR+\frac{2(d-4)}{(d-2)(d-3)}g^{ab}\nabla^2R\\ &\left.+4\frac{(d-5)}{d-3}R^{acbd}R_{cd}-4\frac{(d-4)}{d-3}\nabla^2R^{ab}-2R^{aecd}R^b{}_{ecd}+4R^a_cR^{bc}\right] \end{aligned} \tag{43}$$

and the equation of motion of X^S becomes

$$\begin{aligned} &\mathcal{K}^S+\lambda_1\left(R\mathcal{K}^S-2R^{ab}\mathcal{K}^S_{ab}\right)+\Lambda\left[\mathcal{K}^S\left(\frac{2}{(d-2)(d-3)}R^2-\frac{4}{d-3}R_{a_1b_1}R^{a_1b_1}\right.\right.\\ &\quad\left.+R_{a_1b_1c_1d_1}R^{a_1b_1c_1d_1}\right)-\frac{8}{(d-2)(d-3)}RR^{ab}\mathcal{K}^S_{ab}\\ &\quad+2\frac{(d-4)}{d-2}\mathcal{K}^S_{ab}\left(\nabla^a\nabla^bR+\nabla^b\nabla^aR\right)+\frac{4(d-4)}{(d-2)(d-3)}\mathcal{K}^S\nabla^2R\\ &\quad+8\frac{(d-5)}{d-3}R^{acbd}R_{cd}\mathcal{K}^S_{ab}-8\frac{(d-4)}{d-3}\mathcal{K}^S_{ab}\nabla^2R^{ab}\\ &\quad\left.-4R^{aecd}R^b{}_{ecd}\mathcal{K}^S_{ab}+8\mathcal{K}^S_{ab}R^a_cR^{bc}\right]=0. \end{aligned} \tag{44}$$

We have checked that the Weyl-squared term does not contribute to the entanglement entropy till $d = 8$.

3.1. The precise form of the hypersurface: An example for strip

Let us compute the hypersurface Eq. (40), for the following form of the solution in the bulk

$$ds^2_{d+1} = G_{MN}\, dx^M\, dx^N = -g_{tt}(r)\, dt^2 + g_{xx}(r)\big(dx_1^2 + \cdots + dx_{d-1}^2\big) + g_{rr}(r)\, dr^2 \tag{45}$$

which gives rise to the following induced metric with the embeddings as

$$\begin{aligned}
&X^t = 0, \qquad X^a = x^a = \sigma^a, \qquad X^r = r(x_1);\\
&ds^2_{d-1} = g_{ab}\, d\sigma^a d\sigma^b = \big[g_{xx}(r) + g_{rr}(r) r'^2\big]\, dx_1^2 + g_{xx}(r)\big(dx_2^2 + \cdots + dx_{d-1}^2\big),
\end{aligned} \tag{46}$$

where $r' = \frac{dr}{dx_1}$. The strip is extended along $0 \leqslant x_1 \leqslant \ell$, $-L/2 \leqslant (x_2, \ldots, x_{d-1}) \leqslant L/2$. The explicit computation of the components of $\mathcal{K}^S_{ab}$ gives

$$\begin{aligned}
\mathcal{K}^{x_1}_{x_1 x_1} &= \frac{r' g_{xx}\partial_r g_{xx} + 2r'^3 g_{rr}\partial_r g_{xx} - r'^3 g_{xx}\partial_r g_{rr} - 2r' r'' g_{xx} g_{rr}}{2g_{xx}(g_{xx} + r'^2 g_{rr})},\\
\mathcal{K}^{x_1}_{x_i x_j} &= \frac{r'\partial_r g_{xx}}{2(g_{xx} + r'^2 g_{rr})}\delta_{x_i x_j},\\
\mathcal{K}^{r}_{x_i x_j} &= -\frac{g_{xx}\partial_r g_{xx}}{2g_{rr}(g_{xx} + r'^2 g_{rr})}\delta_{x_i x_j} \quad (i, j = 2, \ldots, d-1)\\
\mathcal{K}^{r}_{x_1 x_1} &= \frac{2g_{rr} g_{xx} r'' - 2r'^2 g_{rr}\partial_r g_{xx} + r'^2 g_{xx}\partial_r g_{rr} - g_{xx}\partial_r g_{xx}}{2g_{rr}(g_{xx} + r'^2 g_{rr})}
\end{aligned} \tag{47}$$

and the rest of the components are zero. For simplicity, let us set the coefficients λ_i's to zero in which case, the extremal hypersurface, $\mathcal{K}^S = 0$, gives

$$2r'' g_{xx} g_{rr} - d r'^2 g_{rr}\partial_r g_{xx} + r'^2 g_{xx}\partial_r g_{rr} - (d-1) g_{xx}\partial_r g_{xx} = 0. \tag{48}$$

Upon using the identity, $r'^3 \frac{d^2 x_1}{dr^2} = -r''$, we can re-write the equation of the hypersurface as

$$\begin{aligned}
&2x_1'' g_{xx} g_{rr} + d g_{rr}\partial_r g_{xx} x_1' - g_{xx}\partial_r g_{rr} x_1' + (d-1) g_{xx}\partial_r g_{xx} x_1'^3 = 0\\
&\Longrightarrow \frac{d}{dr}\left(\frac{g_{xx}^{\frac{d}{2}} x_1'}{\sqrt{g_{rr} + g_{xx} x_1'^2}}\right) = 0,
\end{aligned} \tag{49}$$

where $x_1' = \frac{dx_1}{dr}$. Essentially, we have re-written a second differential equation as a first order differential equation. Now, we can solve the equation of motion and

$$\frac{dx_1}{dr} = \frac{g_{xx}^{\frac{d-1}{2}}(r_\star)\sqrt{g_{rr}(r)}}{\sqrt{g^d_{xx}(r) - g_{xx}(r) g^{d-1}_{xx}(r_\star)}}. \tag{50}$$

In order to determine the constant of integration, we have used the following boundary condition, $x_1'(r_\star) \to \infty$. To get a feel of the solution, let us consider a spacetime that exhibits the scale violating behavior along with the trivial and non-trivial scaling of the spatial direction [28], namely,

$$ds_{d+1}^2 = r^{2\gamma}\left[-\frac{dt^2}{r^{2z}} + \frac{dx_1^2 + \cdots + dx_{d-1}^2}{r^{2\delta}} + \frac{dr^2}{r^2}\right], \quad \text{where } \delta = 0 \text{ or } 1 \text{ with } \gamma \leqslant 0. \tag{51}$$

In which case the differential equation can be exactly solved

$$\pm x_1(r) = c_1 + \frac{r^{d\delta-\gamma(d-1)}}{r_\star^{(d-1)(\delta-\gamma)}} \frac{1}{[d\delta - \gamma(d-1)]} \times {}_2F_1\left[\frac{1}{2}, \frac{d\delta - \gamma(d-1)}{2(d-1)(\delta-\gamma)}, \frac{\delta(3d-2) - \gamma(d-1)}{2(d-1)(\delta-\gamma)}, \left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right], \tag{52}$$

where c_1 is a constant of integration and ${}_2F_1[a, b, c, x]$ is the hypergeometric function. The precise form of c_1 is determined by imposing the boundary condition that $x_1(r = r_\star) = 0$ [22], which gives

$$c_1 = -r_\star^\delta \frac{\sqrt{\pi}}{\delta} \frac{\Gamma(\frac{d\delta-\gamma(d-1)}{2(d-1)(\delta-\gamma)})}{\Gamma(\frac{\delta}{2(d-1)(\delta-\gamma)})}. \tag{53}$$

We can relate ℓ with the constant of integration, c_1, as $\ell/2 = -c_1$. It is easy to see that in the $\delta = 1$, $\gamma = 0$ limit, it re-produces the result of [2]. However, in the $\delta = 0$ and $\gamma = 0$ limit, the solution becomes

$$\pm x_1(r) = c_1 - \frac{1}{\gamma(d-1)} \mathrm{Sin}^{-1}\left(\frac{r_\star}{r}\right)^{\gamma(d-1)}, \qquad c_1 = \frac{\pi}{2\gamma(d-1)}. \tag{54}$$

The entanglement entropy for a generic diagonal and rotationally invariant metric with the entangling region as a strip, $0 \leqslant x_1 \leqslant \ell$, $-L/2 \leqslant (x_2, \ldots, x_{d-1}) \leqslant L/2$, takes the following form [28]

$$2G_N S_{EE} = L^{d-2} \int_\epsilon^{r_\star} dr \frac{\sqrt{g_{xx}^{d-2}(r) g_{rr}(r)}}{\sqrt{1 - (\frac{g_{xx}(r_\star)}{g_{xx}(r)})^{d-1}}} = L^{d-2} \int_\epsilon^{r_\star} dr \frac{r^{(\gamma-\delta)(d-2)+\gamma-1}}{\sqrt{1 - (\frac{r_\star}{r})^{2(\gamma-\delta)(d-1)}}}, \tag{55}$$

where in the second equality we have substituted the geometry as written in Eq. (51). Now, we shall give results to this integral in two different cases i.e., $\delta = 0, 1$ and $\gamma \neq \delta$.

For $\delta = 0$, $\gamma \neq 0$ case: In this case the entanglement entropy gives the following result

$$2G_N S_{EE}(\delta = 0) = -L^{d-2} \frac{\epsilon^{\gamma(d-1)}}{\gamma(d-1)} \sqrt{1 - \left(\frac{r_\star}{\epsilon}\right)^{2\gamma(d-1)}}. \tag{56}$$

It looks from this expression as if the area is a complex quantity but it is not because γ is negative. The entanglement entropy is completely divergent and the divergence goes as $\epsilon^{-|\gamma|(d-1)}$.

For $\delta = 1$, $\gamma \neq 0$ case: In this case the entanglement entropy gives the following result

$$2G_N S_{EE}(\delta = 1) = \left(\frac{r^{1+(d-1)(\gamma-1)}}{1+(d-1)(\gamma-1)} {}_2F_1\left[a, b, c, \left(\frac{r_\star}{r}\right)^{2(d-1)(\gamma-1)}\right]\right)_\epsilon^{r_\star}, \quad \text{for } \gamma \neq \frac{d-2}{d-1} \tag{57}$$

where $a=\frac{1}{2}$, $b=-\frac{1+(d-1)(\gamma-1)}{2(d-1)(\gamma-1)}$, $c=1-\frac{1+(d-1)(\gamma-1)}{2(d-1)(\gamma-1)}$. The dependence on the ϵ goes as $\epsilon^{2-d+\gamma(d-1)}$. It is easy to notice that in the $\gamma\to 0$ limit, it reduces to that written in Eq. (A.10) and gives the precise entanglement entropy as obtained in [2] and the ϵ^{2-d} behavior.

For $\gamma=\frac{d-2}{d-1}$, it is easy to see from Eq. (55) using Eq. (A.22) that there arises a logarithmic dependence of the entanglement entropy as obtained in [26]. For completeness, it comes out as

$$2G_N S_{EE}(\delta=1)=L^{d-2}\int_{\epsilon}^{r_\star} dr\,\frac{r^{(\gamma-1)(d-2)+\gamma-1}}{\sqrt{1-(\frac{r_\star}{r})^{2(\gamma-1)(d-1)}}}\simeq L^{d-2}\,\mathrm{Log}\left(\frac{2r_\star}{\epsilon}\right). \tag{58}$$

Hypersurface at finite coupling but for $\lambda_2=\lambda_3=\lambda_4=0$: Now, let us include the effect of the finite coupling, λ_1, for the diagonal $d+1$ dimensional bulk spacetime, which is AdS. Upon doing a tedious but straight forward calculation, we find the following expressions

$$\begin{aligned}
\mathcal{K}^1 &= \frac{r'[(d-1)g_{xx}g'_{xx}+dr'^2g_{rr}g'_{xx}-r'^2g_{xx}g'_{rr}-2r''g_{xx}g_{rr}]}{2g_{xx}(g_{xx}+g_{rr}r'^2)^2}\\
\mathcal{K}^r &= \frac{2r''g_{xx}g_{rr}-dr'^2g_{rr}g'_{xx}+r'^2g_{xx}g'_{rr}-(d-1)g_{xx}g'_{xx}}{2g_{rr}(g_{xx}+g_{rr}r'^2)^2}\\
R_{x_1x_1} &= \frac{(d-2)}{4g_{xx}^2(g_{xx}+g_{rr}r'^2)}\big[2r'^2g_{xx}g'^2_{xx}+r'^4g_{xx}g'_{xx}g'_{rr}-2r''g^2_{xx}g'_{xx}-2r'^2g^2_{xx}g''_{xx}\\
&\quad -2r'^4g_{xx}g_{rr}g''_{xx}+r'^4g_{rr}g'^2_{xx}\big]\\
R_{x_ix_j} &= \frac{\delta_{x_ix_j}}{4g_{xx}(g_{xx}+g_{rr}r'^2)^2}\big[g_{xx}\big(r'^4g'_{xx}g'_{rr}-(d-5)r'^2g'^2_{xx}-2r''g_{xx}g'_{xx}\\
&\quad -2r'^2g_{xx}g''_{xx}-2r'^4g_{rr}g''_{xx}\big)-(d-4)r'^4g_{rr}g'^2_{xx}\big]\\
R &= \frac{(d-2)}{4g^2_{xx}(g_{xx}+g_{rr}r'^2)^2}\big[2r'^4g_{xx}g'_{xx}g'_{rr}-(d-7)r'^2g_{xx}g'^2_{xx}-4r''g^2_{xx}g'_{xx}\\
&\quad -4r'^2g^2_{xx}g''_{xx}-4r'^4g_{xx}g_{rr}g''_{xx}-(d-5)r'^4g_{rr}g'^2_{xx}\big],
\end{aligned} \tag{59}$$

where $g'_{ij}=\partial_r g_{ij}$ and $r'=\frac{dr}{dx_1}$. In which case, the equation of motion becomes

$$\begin{aligned}
&4g^2_{xx}\big(g_{rr}+g_{xx}x'^2_1\big)\big[-g_{xx}(\partial_r g_{rr})x'_1+dg_{rr}(\partial_r g_{xx})x'_1+(d-1)g_{xx}(\partial_r g_{xx})x'^3_1\\
&\quad +2g_{rr}g_{xx}x''_1\big]+\lambda_1(d-3)(d-2)(\partial_r g_{xx})\big[3g_{xx}(\partial_r g_{rr})(\partial_r g_{xx})x'_1\\
&\quad -(d-4)g_{rr}(\partial_r g_{xx})^2x'_1-(d-7)g_{xx}(\partial_r g_{xx})^2x'^3_1-4g_{rr}g_{xx}\big(\partial^2_r g_{xx}\big)x'_1\\
&\quad -4g^2_{xx}\big(\partial^2_r g_{xx}\big)x'^3_1-2g_{rr}g_{xx}(\partial_r g_{xx})x''_1+4g^2_{xx}(\partial_r g_{xx})x'^2_1x''_1\big]=0,
\end{aligned} \tag{60}$$

where we have included the contribution only from the first two terms of Eq. (36), i.e., have set $\lambda_2=\lambda_3=\lambda_4=0$. It is easy to notice that for $d=2$ the contribution from the induced Ricci scalar vanishes identically. The above equation of motion can be re-written as

$$\frac{d}{dr}\left[g_{xx}^{\frac{d-4}{2}}x'_1\left(\frac{4g_{rr}g^2_{xx}+4g^3_{xx}x'^2_1-\lambda_1(d-2)(d-3)(\partial_r g_{xx})^2}{4(g_{rr}+g_{xx}x'^2_1)^{\frac{3}{2}}}\right)\right]=0. \tag{61}$$

This gives a cubic equation in $x_1'^2$ and all the solution of it are not real. In fact, the real solution for x_1' is a huge expression and finding the exact analytical solution of the hypersurface, $x_1(r)$, is a daunting task. However, the derivative of the function, $x_1(r)$, which is real given in Appendix B.

On computation of the holographic entanglement entropy functional, we find the entanglement entropy takes the following form

$$\begin{aligned}2G_N S_{EE} = L^{d-2}\int dr\,\frac{g_{xx}^{\frac{d-6}{2}}}{4[g_{rr}+g_{xx}x_1'^2]^{\frac{3}{2}}}\big[4g_{xx}^2\big(g_{rr}+g_{xx}x_1'^2\big)^2+\lambda_1(d-2)\big(2g_{xx}g'_{xx}g'_{rr}\\ -(d-7)x_1'^2g_{xx}g_{xx}'^2+4x_1'x_1''g_{xx}^2g'_{xx}-4x_1'^2g_{xx}^2g''_{xx}-4g_{xx}g_{rr}g''_{xx}\\ -(d-5)g_{rr}g_{xx}'^2\big)\big]\end{aligned}\tag{62}$$

Substituting the solution of $x_1'(r)$ to quadratic order in λ_1 from Appendix B into the above expression of the entanglement entropy, gives

$$\begin{aligned}2G_N S_{EE} = {}& L^{d-2}\int dr\,\frac{\sqrt{g_{rr}(r)g_{xx}^{d-2}(r)}}{\sqrt{1-\frac{c^2}{g_{xx}^{d-1}(r)}}}-L^{d-2}\lambda_1\frac{(d-2)}{4}\int dr\,\big[2c^2g_{xx}^2(r)g'_{rr}(r)g'_{xx}(r)\\ &-2g_{xx}^{d+1}(r)g'_{rr}(r)g'_{xx}(r)+6c^2g_{xx}(r)g_{rr}(r)g_{xx}'^2(r)\\ &+(d-5)g_{rr}(r)g_{xx}^{d}(r)g_{xx}'^2(r)+4g_{rr}(r)g_{xx}^{d+1}(r)g''_{xx}(r)\\ &-4c^2g_{rr}(r)g_{xx}^2(r)g''_{xx}(r)\big]\Big/\left[g_{rr}^{3/2}(r)g_{xx}^{\frac{d+6}{2}}(r)\sqrt{1-\frac{c^2}{g_{xx}^{d-1}(r)}}\right]\\ &+c^2\frac{(d-2)^2(d-3)}{32}L^{d-2}\lambda_1^2\int dr\,g_{xx}'^2\big[g_{rr}(r)g_{xx}(r)g_{xx}'^2(r)\big(2c^2(3d+13)\\ &-3(d+9)g_{xx}^{d-1}(r)\big)+12g_{xx}^2\big(c^2-g_{xx}^{d-1}(r)\big)\big(g'_{rr}(r)g'_{xx}(r)\\ &-2g_{rr}(r)g''_{xx}(r)\big)\big]\Big/\left[g_{rr}^{5/2}(r)g_{xx}^{\frac{3d+8}{2}}(r)\sqrt{1-\frac{c^2}{g_{xx}^{d-1}(r)}}\right]+\mathcal{O}(\lambda_1)^3\end{aligned}\tag{63}$$

3.2. Entanglement entropy

We can get the exact expression of the entanglement entropy upon substituting the solution from Appendix B into Eq. (62) for the geometry Eq. (51). Since, the form of $x_1'(r)$ is messy, let us use the leading order (in λ_1) form of it and perform the computation of the entanglement entropy. In which case

$$\begin{aligned}2G_N S_{EE} = {}& L^{d-2}\int_\epsilon^{r_\star} dr\,\frac{r^{(d-1)\gamma-1-(d-2)\delta}}{\sqrt{1-(\frac{r}{r_\star})^{2(d-1)(\delta-\gamma)}}}-\lambda_1L^{d-2}(d-2)\int_\epsilon^{r_\star}\frac{dr}{\sqrt{1-(\frac{r}{r_\star})^{2(d-1)(\delta-\gamma)}}}\\ &\times\bigg[(\gamma-\delta)\big[\gamma(d-3)-(d-1)\delta\big]r^{-1+(d-3)\gamma-(d-2)\delta}\\ &+2(\gamma-\delta)(2\gamma-\delta)\frac{r^{-1-\gamma(d+1)+d\delta}}{r_\star^{2(d-1)(\delta-\gamma)}}\bigg]\end{aligned}$$

$$
\begin{aligned}
&-\lambda_1^2(d-2)^2(d-3)(\gamma-\delta)^3L^{d-2}\int_{\epsilon}^{r_\star}\frac{dr}{\sqrt{1-(\frac{r}{r_\star})^{2(d-1)(\delta-\gamma)}}}\\
&\times\left[\frac{3}{2}\big[\gamma(d+5)-\delta(d+1)\big]\frac{r^{d\delta-1-\gamma(d+3)}}{r_\star^{2(d-1)(\delta-\gamma)}}\right.\\
&\left.-\big[\gamma(3d+7)-\delta(3d+1)\big]\frac{r^{(3d-2)\delta-1-\gamma(3d+1)}}{r_\star^{4(d-1)(\delta-\gamma)}}\right]+\cdots,
\end{aligned}
\tag{64}
$$

where the ellipses stands for higher order terms in λ_1. On performing the integrals results in

$$
\begin{aligned}
2G_N S_{EE}={}&\frac{L^{d-2}}{[\gamma(d-1)-\delta(d-2)]}\left(r^{\gamma(d-1)-\delta(d-2)}{}_2F_1\left[\frac{1}{2},\frac{\gamma(d-1)-\delta(d-2)}{2(d-1)(\delta-\gamma)},\right.\right.\\
&\left.\left.\frac{d\delta-\gamma(d-1)}{2(d-1)(\delta-\gamma)},\left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right]\right)_{\epsilon}^{r_\star}\\
&-\lambda_1L^{d-2}\frac{(d-2)(\gamma-\delta)}{[\gamma(d-3)-(d-2)\delta]}\big[\gamma(d-3)-(d-1)\delta\big]\\
&\times\left(r^{(d-3)\gamma-\delta(d-2)}{}_2F_1\left[\frac{1}{2},\frac{\gamma(d-3)-(d-2)\delta}{2(d-1)(\delta-\gamma)},\right.\right.\\
&\left.\left.\frac{d\delta-\gamma(d+1)}{2(d-1)(\delta-\gamma)},\left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right]\right)_{\epsilon}^{r_\star}\\
&+\lambda_1L^{d-2}\frac{2(d-2)(2\gamma-\delta)(\gamma-\delta)}{[\gamma(d+1)-d\delta]}r_\star^{2(\gamma-\delta)(d-1)}\\
&\times\left(r^{d\delta-\gamma(d+1)}{}_2F_1\left[\frac{1}{2},\frac{d\delta-\gamma(d+1)}{2(d-1)(\delta-\gamma)},\right.\right.\\
&\left.\left.1+\frac{d\delta-\gamma(d+1)}{2(d-1)(\delta-\gamma)},\left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right]\right)_{\epsilon}^{r_\star}\\
&+\lambda_1^2(d-2)^2\frac{3}{2}(d-3)L^{d-2}(\gamma-\delta)^3\frac{[\gamma(d+5)-\delta(d+1)]}{\gamma(d+3)-d\delta}r_\star^{2(\gamma-\delta)(d-1)}\\
&\times\left(r^{d\delta-\gamma(d+3)}{}_2F_1\left[\frac{1}{2},\frac{d\delta-\gamma(d+3)}{2(d-1)(\delta-\gamma)},\right.\right.\\
&\left.\left.\frac{(3d-2)\delta-\gamma(3d+1)}{2(d-1)(\delta-\gamma)},\left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right]\right)_{\epsilon}^{r_\star}\\
&-\lambda_1^2(d-2)^2(d-3)L^{d-2}(\gamma-\delta)^3\left[\frac{\gamma(3d+7)-\delta(3d+1)}{\gamma(3d+1)-(3d-2)\delta}\right]r_\star^{4(\gamma-\delta)(d-1)}\\
&\times\left(r^{(3d-2)\delta-\gamma(3d+1)}{}_2F_1\left[\frac{1}{2},\frac{(3d-2)\delta-\gamma(3d+1)}{2(d-1)(\delta-\gamma)},\right.\right.\\
&\left.\left.\frac{(5d-4)\delta-\gamma(5d-1)}{2(d-1)(\delta-\gamma)},\left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right]\right)_{\epsilon}^{r_\star}+\cdots.
\end{aligned}
\tag{65}
$$

It is worth mentioning that till this order in λ_1 do not give any logarithmic violation of the entanglement entropy for any choice of γ with δ either 0 or 1 except $\gamma=\frac{d-2}{d-1}$ with $\delta=1$.

To get a feel of the entanglement entropy for AdS solution, which means setting $\delta = 1$ and $\gamma = 0$, restoring the AdS radius R_0

$$2G_N S_{EE} = \frac{L^{d-2} R_0^{d-1}}{\epsilon^{d-2}} \left(\frac{1-(d-1)(d-2)(\lambda_1/R_0^2)}{(d-2)} \right) - L^{d-2} R_0^{d-1} \frac{\sqrt{\pi}\,\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})} r_\star^{2-d} \left(\frac{1+(d-2)(d-3)(\lambda_1/R_0^2)}{d-2} \right) + \mathcal{O}(\lambda_1)^2, \tag{66}$$

where we have taken the $\epsilon \to 0$ limit and kept both the divergent and finite terms. In the absence of the higher derivative correction, the UV divergence was found to be ϵ^{2-d} in $d+1$ dimensional bulk spacetime [2]. And, upon inclusion of the next higher derivative term to the holographic entanglement entropy functional gives the same power of the UV divergent, ϵ^{2-d}. This is observed, however only, for $d = 4$ in [17].

We can re-express the above expression of the entanglement entropy in terms of the size ℓ and R. In which case, it reads as

$$2G_N S_{EE} = \frac{L^{d-2} R^{d-1}}{4\epsilon^{d-2}} \left(\frac{4-(d+1)(d-1)(d-2)a\lambda_a}{(d-2)} \right) - 2^{d-2} L^{d-2} R^{d-1} \pi^{\frac{d-1}{2}} \left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})} \right)^{d-1} \times \ell^{2-d} \left(\frac{1+\frac{3}{4}(d-1)(d-2)(d-3)a\lambda_a}{d-2} \right) + \mathcal{O}(\lambda_a)^2. \tag{67}$$

Note that R_0 and R are the radii of AdS spacetime with and without the higher derivative correction, whose precise relation is given in Section 6 and $\lambda_1 = a\lambda_a R^2$. Upon comparing Eq. (36) with Eq. (C.24) of [18], we find $a = \frac{2}{(d-2)(d-3)}$, after identifying the couplings $\lambda_a = \lambda_{there}$.

It is easy to see that up to linear in the coupling, λ_a, the finite part and the singular part of the entanglement entropy obeys following differential equations

$$\frac{d}{d\ell}\left(\ell^{d-2} S_{EE}^{fp}\right) = 0, \qquad \frac{d^{d-1}}{d\ell^{d-1}}\left(\ell^{d-2} S_{EE}^{sp}\right) = 0, \tag{68}$$

where S_{EE}^{fp} and S_{EE}^{sp} stand for the finite part and singular part of the entanglement entropy, respectively. However, there exists another differential equation for the full S_{EE}

$$\ell \frac{\partial}{\partial \ell}\left(\ell \frac{\partial}{\partial \ell} + (d-2) \right) S_{EE} = 0. \tag{69}$$

Let us re-write the expression of the entanglement entropy as

$$2G_N S_{EE} = \frac{L^{d-2} R^{d-1}}{\epsilon^{d-2}} \left(\frac{2d-6-(d+1)(d-1)\lambda_a}{2(d-2)(d-3)} \right) - 2^{d-2} L^{d-2} R^{d-1} \pi^{\frac{d-1}{2}} \left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})} \right)^{d-1} \ell^{2-d} \left(\frac{1+\frac{3}{2}(d-1)\lambda_a}{d-2} \right) + \mathcal{O}(\lambda_a)^2 \tag{70}$$

We can check that the quantities defined in Eq. (12) holds good for $d=4$ when we take the coupling to stay within the following window $-\frac{7}{36} \leqslant \lambda_a \leqslant \frac{9}{100}$. This bound follows from the study of causality and the positivity of the energy flux in [29–31].

3.3. Gauss–Bonnet combination

In this section, we shall find the form of the extremal hypersurface as well as the entanglement entropy to leading order in the coupling λ_i for a very special combination of the λ_2, λ_3 and λ_4. In which case the holographic entanglement entropy functional takes the following form as in [18]

$$4G_N S_{EE} = \int d^{d-1}\sigma \sqrt{\det(g_{ab})}\big[1+\lambda_1 R(g) + \Lambda\big(R^2(g) - 4R_{ab}(g)R^{ab}(g) + R_{abcd}(g)R^{abcd}(g)\big)\big]. \tag{71}$$

In what follows, we shall be interested to calculate the entanglement entropy when the entangling region is of the strip type. In which case, the induced geometry is as written in Eq. (46). For simplicity of doing the computation, we shall fix the dimensionality of the bulk spacetime.

For $d=5$: The bulk spacetime is a $5+1$ dimensional system whereas the induced metric is a 4 dimensional spatial metric. In this case, the Gauss–Bonnet combination, $R^2(g) - 4R_{ab}(g)R^{ab}(g) + R_{abcd}(g)R^{abcd}(g)$, is non-zero but topological. It gives a non-zero contribution to the action but not to the equation of motion of the embedding field. This also agrees with the computation of the equation of motion of the embedding field, X^M, following from Eq. (42). In this case, the equation of motion can also be obtained from Eq. (60) by considering $d=5$. Hence, it is easy to conclude that the form of the hypersurface is the same as in the previous case. Moreover, the holographic entanglement entropy becomes

$$\begin{aligned}
&2G_N S_{EE} \\
&\quad = L^3 \int dr\, g_{xx}^{3/2}\sqrt{g_{rr}+g_{xx}x_1'^2}\bigg[1+3\lambda_1 \\
&\qquad \times \left(\frac{g'_{rr}g'_{xx} + g'^2_{xx}x_1'^2 - 2(g_{rr}+g_{xx}x_1'^2)g''_{xx} + 2g_{xx}g'_{xx}x'_1x''_1}{2g_{xx}(g_{rr}+g_{xx}x_1'^2)^2}\right) \\
&\qquad + 3\Lambda g'^2_{xx}\left(\frac{g_{rr}g'^2_{xx} + g_{xx}(g'_{rr}g'_{xx} + 2g'^2_{xx}x_1'^2 - 2g_{rr}g''_{xx}) + 2g^2_{xx}x'_1(g'_{xx}x''_1 - x'_1g''_{xx})}{2g^4_{xx}(g_{rr}+g_{xx}x_1'^2)^3}\right)\bigg]
\end{aligned} \tag{72}$$

Substituting the solution, x_1', from Appendix B, for the AdS geometry and doing the r integral resulting in the entanglement entropy to linear order in λ_1 and Λ

$$2G_N S_{EE} = L^3 R_0^4\left(\frac{1-\frac{12\lambda_1}{R_0^2}+\frac{24\Lambda}{R_0^4}}{3\epsilon^3}\right) - 11R_0^4\sqrt{\pi}L^3\frac{(1+\frac{6\lambda_1}{R_0^2})}{64r_\star^3}\frac{\Gamma(\frac{-11}{8})}{\Gamma(\frac{1}{8})}, \tag{73}$$

where ϵ and $r_\star$ are the UV regulator and the point of maximum extension along the r direction, respectively. It is interesting to observe that to the linear order in Λ, the Gauss–Bonnet coefficient does not enter in the finite term of S_{EE} whereas it enters in the divergent piece.

For $d = 6$: For this case, with the induced metric as written in Eq. (46) gives the following holographic entanglement entropy functional

$$
\begin{aligned}
&2G_N S_{EE}\\
&= L^4 \int dr\, g_{xx}^2 \sqrt{g_{rr} + g_{xx} x_1'^2} \bigg[1 + \lambda_1 \\
&\quad \times \left(\frac{-g_{rr} g_{xx}'^2 + g_{xx}(x_1'^2 g_{xx}'^2 + 2 g_{rr}' g_{xx}' - 4 g_{rr} g_{xx}'') + 4 g_{xx}^2 x_1' (g_{xx}' x_1'' - x_1' g_{xx}''}{g_{xx}^2 (g_{rr} + g_{xx} x_1'^2)^2} \right) \\
&\quad - 3\Lambda g_{xx}'^2 \left(\frac{3 g_{rr} g_{xx}'^2 + g_{xx}(4 g_{rr}' g_{xx}' + 7 g_{xx}'^2 x_1'^2 - 8 g_{rr} g_{xx}'') + 8 g_{xx}^2 x_1' (g_{xx}' x_1'' - x_1' g_{xx}'')}{2 g_{xx}^4 (g_{rr} + g_{xx} x_1'^2)^3} \right). \bigg]
\end{aligned}
\tag{74}
$$

The equation of motion that follows takes the following form

$$
\begin{aligned}
&2 g_{xx}^4 \big(g_{rr} + g_{xx} x_1'^2\big)^2 \big(g_{xx} g_{rr}' x_1' - 6 g_{rr} g_{xx}' x_1' - 5 g_{xx} g_{xx}' x'^3 - 2 g_{rr} g_{xx} x''\big) \\
&\quad + 6\lambda_1 g_{xx}^2 g_{xx}' \big(g_{rr} + g_{xx} x_1'^2\big)\big(-3 g_{xx} g_{rr}' g_{xx}' x_1' + 2 g_{rr} g_{xx}'^2 x_1' - g_{xx} g_{xx}'^2 x'^3 \\
&\quad + 4 g_{rr} g_{xx} g_{xx}'' x_1' + 4 g_{xx}^2 g_{xx}'' x'^3 + 2 g_{rr} g_{xx} g_{xx}' x'' - 4 g_{xx}^2 g_{xx}' x_1'^2 x_1''\big) \\
&\quad + 3\Lambda g_{xx}'^3 \big(5 g_{xx} g_{rr}' g_{xx}' x_1' + 2 g_{rr} g_{xx}'^2 x_1' + 7 g_{xx} g_{xx}'^2 x_1'^3 - 8 g_{rr} g_{xx} g_{xx}'' x_1' - 8 g_{xx}^2 g_{xx}'' x_1'^3 \\
&\quad - 2 g_{rr} g_{xx} g_{xx}' x_1'' + 8 g_{xx}^2 g_{xx}' x_1'^2 x_1''\big) = 0.
\end{aligned}
\tag{75}
$$

This equation of motion can be re-written as

$$
\frac{d}{dr}\left[\frac{x_1' g_{xx}^3}{\sqrt{g_{rr} + g_{xx} x_1'^2}} - 3\lambda_1 \frac{g_{xx} g_{xx}'^2 x_1'}{(g_{rr} + g_{xx} x_1'^2)^{3/2}} + 3\Lambda \frac{g_{xx}'^4 x_1'}{2 g_{xx} (g_{rr} + g_{xx} x_1'^2)^{5/2}} \right] = 0. \tag{76}
$$

Upon solving the equation of motion, we find to linear order in λ_1 and Λ as

$$
x_1'(r) = \frac{c\sqrt{g_{rr}}}{\sqrt{g_{xx}^6 - c^2 g_{xx}}} + \frac{3c\lambda_1 g_{xx}'^2}{g_{xx}^2 \sqrt{g_{rr}} \sqrt{g_{xx}^6 - c^2 g_{xx}}} + 3c\Lambda \frac{(c^2 - g_{xx}^5) g_{xx}'^4}{2 g_{rr}^{3/2} g_{xx}^9 \sqrt{g_{xx}^6 - c^2 g_{xx}}}, \tag{77}
$$

where c is a constant of integration and is determined by demanding that as $r \to r_\star$, $x_1'(r_\star)$ diverges. Substituting this form of the solution into the action, results in

$$
\begin{aligned}
2G_N S_{EE} = L^4 \int dr \bigg[&\frac{\sqrt{g_{rr}} g_{xx}^{9/2}}{\sqrt{g_{xx}^5 - c^2}} \\
&- \lambda_1 \frac{6c^2 g_{rr} g_{xx}'^2 + g_{rr} g_{xx}^5 g_{xx}'^2 + 2 g_{xx} (c^2 - g_{xx}^5)(g_{rr}' g_{xx}' - 2 g_{rr} g_{xx}'')}{g_{rr}^{3/2} g_{xx}^{5/2} \sqrt{g_{xx}^5 - c^2}} \\
&- \Lambda \frac{3 g_{xx}'^2 \sqrt{g_{xx}^5 - c^2}}{2 g_{rr}^{5/2} g_{xx}^{19/2}} \big(g_{rr} g_{xx}'^2 \big(3 g_{xx}^5 - 22c^2\big) \\
&- 4 g_{xx} \big(c^2 - g_{xx}^5\big)\big(g_{rr}' g_{xx}' - 2 g_{rr} g_{xx}''\big)\big) \bigg]
\end{aligned}
\tag{78}
$$

Let us use the geometry of AdS spacetime and do the r integral from ϵ, the UV cutoff, to the maximum extension in IR, $r_\star$. This gives

$$2G_N S_{EE} = \frac{L^4 R_0^5}{4\epsilon^4}\left(1 - \frac{20\lambda_1}{R_0^2} + \frac{120\Lambda}{R_0^4}\right) - \frac{L^4 R_0^5}{44 r_\star^4}\sqrt{\pi}\left(11 + \frac{132\lambda_1}{R_0^2} - \frac{120\Lambda}{R_0^4}\right)\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{10})}, \tag{79}$$

where we have set the constant $c = R_0^5/r_\star^5$. In our previous studies, we found that the divergent term to the entanglement entropy goes as ϵ^{2-d} and the finite term has the following dependence, $r_\star^{2-d}$. For the Gauss–Bonnet term in the holographic entanglement entropy functional, we found this behavior again.

Now computing the size ℓ from Eq. (77) to linear order in the couplings

$$\frac{\ell}{2} = \sqrt{\pi}\left(11 + \frac{132\lambda_1}{R_0^2} - \frac{120\Lambda}{R_0^4}\right)\frac{\Gamma(\frac{8}{5})}{66\Gamma(\frac{11}{10})} r_\star + \cdots, \tag{80}$$

where the ellipses stand for the terms higher order in the couplings. Re-expressing the entanglement entropy in terms of the size ℓ

$$2G_N S_{EE} = \frac{L^4 R_0^5}{4\epsilon^4}\left(1 - \frac{20\lambda_1}{R_0^2} + \frac{120\Lambda}{R_0^4}\right) - \frac{4L^4 R_0^5}{11\ell^4}\pi^{5/2}\left(11 + \frac{660\lambda_1}{R_0^2} - \frac{600\Lambda}{R_0^4}\right)\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{10})}\right)^5 + \cdots. \tag{81}$$

Using the following relation $R_0 = R(1 - 3a\lambda_a - 12X_2\Lambda_a)$, we can re-write

$$2G_N S_{EE} = \frac{L^4 R^5}{4\epsilon^4}(1 - 35a\lambda_a + 300X_2\Lambda_a) - \frac{4L^4 R^5}{11\ell^4}\pi^{5/2}(11 + 495a\lambda_a - 2460X_2\Lambda_a)\left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{10})}\right)^5 + \cdots, \tag{82}$$

where we have set $\lambda_1 = a\lambda_a R^2$ and $\Lambda = 3X_2 R^4\Lambda_a$. Upon comparing Eq. (36) with Eq. (C.24) of [18], we find $a = 1/6$ and $X_2 = -1/24$ for $d = 6$, after identifying the coupling $\lambda_a = \lambda_{there}$ and $\Lambda = \mu_{there}$. From this expression of the entanglement entropy, it is easy to notice that S_{EE} obeys the following differential equation for $d = 6$

$$\ell\frac{\partial}{\partial\ell}\left(\ell\frac{\partial}{\partial\ell} + (d-2)\right)S_{EE} = 0. \tag{83}$$

4. Higher derivative corrected extremal surfaces can't penetrate the horizon

The higher derivative corrected equation of motion with $\lambda_2 = \lambda_3 = \lambda_4 = 0$ in any arbitrary dimension can be re-written as

$$4g_{xx}^2\left(g_{xx} + g_{rr}r'^2\right)\left[-(d-1)g_{xx}(\partial_r g_{xx}) + g_{xx}(\partial_r g_{rr})r'^2 - dg_{rr}(\partial_r g_{xx})r'^2 + 2g_{rr}g_{xx}r''\right] + \lambda_1(d-3)(d-2)(\partial_r g_{xx})\left[-3g_{xx}(\partial_r g_{rr})(\partial_r g_{xx})r'^4\right.$$

$$+ (d-4)g_{rr}(\partial_r g_{xx})^2 r'^4 + (d-7)g_{xx}(\partial_r g_{xx})^2 r'^2 + 4g_{rr}g_{xx}\big(\partial_r^2 g_{xx}\big)r'^4$$
$$+ 4g_{xx}^2\big(\partial_r^2 g_{xx}\big)r'^2 - 2g_{rr}g_{xx}(\partial_r g_{xx})r'^2 r'' + 4g_{xx}^2(\partial_r g_{xx})r''\big] = 0. \quad (84)$$

Now we shall set up our logic following [22] and show that after the inclusion of the higher derivative terms, i.e., with $\lambda_1 \neq 0$, the extremal surfaces can't penetrate the horizon, r_h. The argument goes as follows. Eq. (84) essentially describes the equation of the hypersurface that ends on the boundary and goes deep into the bulk. Let us assume that it goes till $r = r_\star$, which is the deepest that it can go and then turns around and ends at the boundary. So at this point, $r_\star$, the derivative of the function r with respect to x_1 vanishes, i.e., $(\frac{dr}{dx_1})_{r_\star} = 0$. Putting this piece of information into Eq. (84) gives

$$\big[g_{xx}^2\big(2g_{rr}r'' - (d-1)\partial_r g_{xx}\big) + \lambda_1(d-2)(d-3)(\partial_r g_{xx})^2 r''\big]_{r_\star} = 0. \quad (85)$$

Before moving onto discuss the $\lambda_1 \neq 0$ case, let us first discuss the $\lambda_1 = 0$ case. Also, we want to make few assumptions on the metric components. Let there be a horizon at $r = r_h$, if there exists more than one then this is the outermost horizon. The quantity g_{rr} changes sign as we go beyond r_h, i.e., $g_{rr} < 0$ for $r > r_h$[10] and assume $\partial_r g_{xx}$ is always negative.[11] On summarizing the assumptions:

$$g_{rr}(r) = \begin{cases} +ve & \text{for } r < r_h \quad \text{(Outside the horizon)} \\ -ve & \text{for } r > r_h \quad \text{(Inside the horizon)} \end{cases}$$

and

$$g_{xx}(r) \text{ is } +ve \text{ for any } r; \qquad \text{whereas } g'_{xx}(r) \text{ is } -ve \text{ for any } r \quad (86)$$

$\lambda_1 = 0$: Let us demand that the extremal surface goes deep into the bulk and has the maximum extension. It means we need to impose the above mentioned condition along with the further condition that the quantity $(\frac{d^2r}{dx_1^2})_{r_\star} < 0$. For vanishing λ_1, there follows from Eq. (85) at $r = r_\star$ that

$$\big[2r'' g_{rr} - (d-1)g'_{xx}\big]_{r_\star} = 0. \quad (87)$$

For $r_\star < r_h$, i.e., the point $r_\star$ is outside the horizon. In which case, we can easily satisfy the above equation. To make things clear $(r'' g_{rr})_{r_\star}$ is $-ve$ whereas $g'_{xx}(r_\star)$ is $-ve$.

For $r_\star > r_h$, i.e., the point $r_\star$ is inside the horizon. In which case, we can't satisfy the above equation. It means that the extremal surfaces cannot penetrate the horizon because the sum of two positive quantities can't give zero. This is the argument put forward in [22].

$\lambda_1 \neq 0$: Once again imposing the condition that the extremal surface goes deep into the bulk and has the maximum extension means we need to put the conditions as mentioned above along with $(\frac{d^2r}{dx_1^2})_{r_\star} < 0$. The corresponding equation for the hypersurface at $r = r_\star$ is given by

$$\big[g_{xx}^2\big(2r'' g_{rr} - (d-1)g'_{xx}\big) + \lambda_1(d-2)(d-3)r'' g'^2_{xx}\big]_{r_\star} = 0. \quad (88)$$

[10] Remember that the boundary is at $r = 0$.

[11] This is true for AdS spacetime.

As the hypersurface goes inside the horizon, $r_\star > r_h$, we can satisfy the above equation provided $\lambda_1 > 0$. It means for positive values of λ_1, the spacelike hypersurfaces can cross the horizon. However, as we shall check explicitly for AdS black hole spacetime the condition $(\frac{d^2r}{dx_1^2})_{r_\star} < 0$ can't be obeyed. Hence, the spacelike hypersurface can't penetrate the horizon.

Let us examine this in detail, at least to linear order in λ_1, using the solution as written in Appendix B. Recall that $\frac{d^2r}{dx_1^2} = -\frac{1}{x_1'^3}\frac{d^2x_1}{dr^2}$. It means $\frac{1}{x_1'^3}\frac{d^2x_1}{dr^2}$ should be $+ve$ at $r = r_\star$ for positive coupling. Let us check this explicitly for the following geometry [34] with unit AdS radius

$$\begin{aligned} g_{xx} &= \frac{1}{r^2}, \qquad g_{rr} = \frac{2\lambda_a}{r^2}\big[1 - \sqrt{1 - 4\lambda_a (r/r_h)^d}\big]^{-1}, \\ c &= \frac{1}{r_\star^{d-1}}, \qquad \lambda_1 = \frac{2}{(d-1)(d-3)}\lambda_a \end{aligned} \tag{89}$$

and it follows that

$$\left(\frac{1}{x_1'^3}\frac{d^2x_1}{dr^2}\right)_{r_\star} = \frac{(d-1)}{r_\star}\left(1 - \frac{r_\star^d}{r_h^d}\right) - 3\lambda_a \frac{(d-1)}{r_\star}\left(1 - \frac{r_\star^d}{r_h^d}\right)^2. \tag{90}$$

Let us ask the question: Can there be an instance for $r_\star \geqslant r_h$ with positive dimension, d, as well as positive coupling, $\lambda_a > 0$, for which the quantity $(\frac{1}{x'^3}\frac{d^2x_1}{dr^2})_{r_\star}$ becomes $+ve$? The answer is none. This suggests that the spacelike hypersurfaces can't penetrate the horizon.

Let us look at the penetration of the hypersurface from the point of view of the solution of the embedding field, $x_1(r)$, from Eq. (B.3) of Appendix B. Using Eq. (89), we find to leading order in λ_a

$$\begin{aligned} x_1(r) &= c_1 + \int \frac{dr}{r_\star^{d-1}\sqrt{1 - (\frac{r}{r_\star})^{2(d-1)}}}\frac{r^{d-1}}{\sqrt{1 - (\frac{r}{r_h})^d}} \\ &\quad + \frac{3\lambda_a}{2}\int dr\, \frac{r_\star^{1-d} r^{d-1}}{\sqrt{1 - (\frac{r}{r_\star})^{2(d-1)}}}\sqrt{1 - \left(\frac{r}{r_h}\right)^d} \\ &= c_1 + r_\star \int \frac{dt}{\sqrt{1 - t^{2(d-1)}}}\frac{t^{d-1}}{\sqrt{1 - (\frac{r_\star}{r_h})^d t^d}} \\ &\quad + \frac{3\lambda_a r_\star}{2}\int dx\, \frac{t^{d-1}}{\sqrt{1 - t^{2(d-1)}}}\sqrt{1 - \left(\frac{r_\star}{r_h}\right)^d t^d} \end{aligned} \tag{91}$$

where we have defined $t \equiv r/r_\star$ and c_1 is the constant of integration. The constant of integration, c_1, is determined by imposing the boundary condition that $x_1(r_\star) = 0$. From Fig. 1 it is easy to notice that $\frac{x_1(r) - c_1}{r_\star}$ is positive, this means using the boundary condition at $r = r_\star$, we get c_1 to be a negative quantity.

One half of the figure of the full U shaped profile is plotted in Fig. 1. Let us note that the hypersurface can go from 0 to $r_\star$ means $0 \leqslant t \leqslant 1$. For $r_\star > r_h$ and t approaching unity means $\sqrt{1 - (\frac{r_\star}{r_h})^d t^d}$ becomes complex. This means inside the horizon there does not exist any real valued solution, which suggests that the hypersurface can not penetrate the horizon.

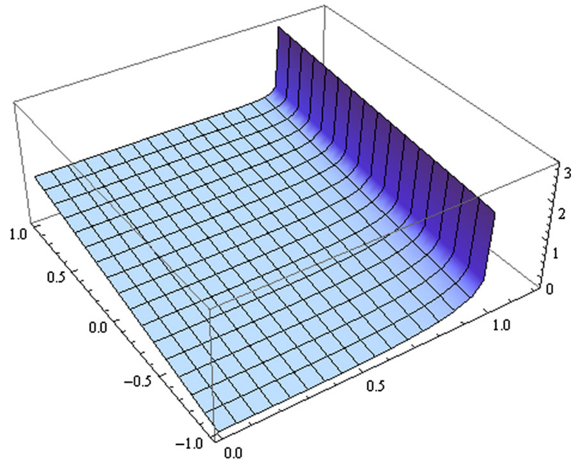

Fig. 1. The graph is plotted for $\frac{x_1(r)-c_1}{r_\star}$ versus $r_\star/r_h$ in $d=4$. The parameters stay in the following range: $-1\leqslant\lambda_a\leqslant 1$ and $0\leqslant r_\star/r_h\leqslant 1.2$.

5. Fluctuating geometry

In this section, we would like to calculate the change in the entanglement entropy due to the background metric fluctuation, i.e., $G_{MN}\to G_{MN}+\delta G_{MN}$ and keeping terms to linear order in δG_{MN}. Under such a change, the induced metric changes as $g_{ab}\to g_{ab}+\delta g_{ab}$, where $\delta g_{ab}=\partial_a X^M\partial_b X^N\delta G_{MN}$. If we set $\lambda_{i+1}=0$ for $i\geqslant 1$, for simplicity, then the change in the entanglement entropy can be calculated from

$$\begin{aligned}4G_N\Delta S_{EE}=\int d^{d-1}\sigma\sqrt{\det g}\bigg[\frac{1}{2}g^{cd}h_{cd}\\&+\lambda_1\bigg(-R^{ab}h_{ab}+\nabla^a\nabla^b h_{ab}-g^{cd}\nabla^2 h_{cd}+\frac{R}{2}g^{cd}h_{cd}\bigg)\bigg],\end{aligned}\tag{92}$$

where $h_{ab}=\delta g_{ab}$. The indices are raised and lowered using g_{ab} and its inverse. ∇_a is defined with respect to g_{ab}. In what follows, we are going to use a particular kind of metric fluctuation, namely that is used in the paper [23], where the fluctuating metric is diagonal and asymptotes to the AdS geometry. Since, we did the computation of the entanglement entropy in full generality for the diagonal form of the metric as in (63), which suggests we can now do the computation for the fluctuating geometry as well.

5.1. One parameter fluctuation with $\Lambda=0$

For completeness, let us write down the complete form of the metric with fluctuations and the bulk cosmological constant, Λ_c

$$\begin{aligned}&g_{tt}=\frac{R_0^2}{r^2}\big(1-mr^d\big),\qquad g_{xx}=\frac{R_0^2}{r^2},\qquad g_{rr}=\frac{R_0^2}{r^2}\big(1+mr^d\big),\\&\Lambda_c=-\frac{(d-1)(d-2)}{2R^2}\end{aligned}\tag{93}$$

where we have restored AdS radius. m is a constant and assumed to be very small, in the sense of [23]. In which case

$$\Delta S_{EE} = S_{EE}(m) - S_{EE}(m=0) = \frac{m(d-1)L^{d-2}R_0^{d-1}}{32G_N\sqrt{\pi}(d+1)}\ell^2 \frac{(\Gamma(\frac{1}{2(d-1)}))^2}{(\Gamma(\frac{d}{2(d-1)}))^2}\frac{\Gamma(\frac{d}{(d-1)})}{\Gamma(\frac{d+1}{2(d-1)})} \times \left[1 - 3a(d-2)(d-3)\frac{R^2}{R_0^2}\lambda_a\right], \tag{94}$$

where we have set $\lambda_1 = aR^2\lambda_a$ with a a constant. Now, we shall move onto the computation of the energy (or mass) of the excited state using the AdS/CFT dictionary as worked out in [33]

$$\Delta M = \int d^{d-1}x\sqrt{\det(\sigma)_{ij}}Nu^M u^N T_{MN}, \tag{95}$$

which in our case using $N = \sqrt{g_{tt}}$, $u^t = 1/\sqrt{g_{tt}}$ and $\sqrt{\det(\sigma)_{ij}} = g_{xx}^{\frac{d-1}{2}} = (R_0/r)^{d-1}$, gives

$$\Delta M = \int d^{d-1}x\left(\frac{R_0}{r}\right)^{d-1}\frac{T_{tt}}{\sqrt{g_{tt}}}. \tag{96}$$

The t–t component of the energy–momentum tensor can be calculated from [32]

$$T_{tt} = \frac{1}{8\pi G_N}\left(K_{tt} + Kg_{tt} + \frac{d-1}{\tilde{R}}g_{tt} + 2\lambda\left[-\frac{KK_{tt}^2}{g_{tt}} - \frac{1}{3}\frac{K_{tt}^3}{g_{tt}^2} + (d-1)\frac{K_{tt}K_{xx}^2}{g_{xx}^2} - K^2K_{tt} + (d-1)\frac{Kg_{tt}K_{xx}^2}{g_{xx}^2} - \frac{g_{tt}}{3}K^3 - \frac{2(d-1)}{3}\frac{g_{tt}K_{xx}^3}{g_{xx}^3}\right]\right), \tag{97}$$

where $K_{tt} = -\frac{g'_{tt}}{2\sqrt{g_{rr}}}$, $K_{xx} = \frac{g'_{xx}}{2\sqrt{g_{rr}}}$ and $K = \frac{g'_{tt}}{2g_{tt}\sqrt{g_{rr}}} + \frac{(d-1)g'_{xx}}{2g_{xx}\sqrt{g_{rr}}}$. The quantity $\tilde{R}$ is defined in such a way that as $\lambda \to 0$, it approaches the size of the AdS spacetime,[12] i.e., $\lim_{\lambda\to 0}\tilde{R} \to R$. It means we can write $\tilde{R} = R + \lambda R_1$ for small λ and we shall determine R_1 by demanding that T_{tt} is not diverging as we approach the boundary, $r \to 0$. The quantity λ is the same as α_2 in the notation of [32].

Substituting all these ingredients into T_{tt} gives

$$T_{tt} = \frac{(d-1)}{16\pi G_N}mR_0r^{d-2} - \frac{(d-1)}{8\pi G_N R_0}m\lambda\left(d^2 - 5d + 6\right)r^{d-2}, \tag{98}$$

where $R_1 = \frac{2(d^2-5d+6)}{3R_0}$. The mass comes as

$$\Delta M = \frac{(d-1)}{16\pi G_N}m\ell L^{d-2}R_0^{d-1} - \frac{(d-1)}{8\pi G_N}\lambda m\ell\left(d^2 - 5d + 6\right)L^{d-2}R_0^{d-3} \tag{99}$$

On comparing with the bulk action and the holographic entanglement entropy functional of [17] and [18] with ours, we find that $2\lambda = \lambda_1$. So the mass of the excited state becomes

$$\Delta M = \frac{(d-1)}{16\pi G_N}m\ell L^{d-2}R_0^{d-1} - \frac{(d-1)}{16\pi G_N}\lambda_1 m\ell\left(d^2 - 5d + 6\right)L^{d-2}R_0^{d-3} \tag{100}$$

[12] We use R to denote the size of AdS spacetime without the higher derivative term, whereas R_0 with higher derivative term. The relation between them can be read out from [34], $R_0^2 = \frac{R^2}{2}[1 + \sqrt{1 - \frac{4(d-2)(d-3)\lambda}{R^2}}]$.

The AdS radius R_0 is related to R as $R_0 = R/\sqrt{f_\infty}$, where f_∞ is found by solving $1 - f_\infty + (a/2)(d-2)(d-3)\lambda_a f_\infty^2 = 0$. To leading order in λ_a, we take $R_0 = R - (a/4)(d-2)(d-3)R\lambda_a$. Finally, taking the ratio of the change in the entanglement with the mass gives

$$\frac{\Delta S_{EE}}{\Delta M} = \frac{\sqrt{\pi}(\Gamma(\frac{1}{2(d-1)}))^2\Gamma(\frac{1}{(d-1)})}{2(d^2-1)(\Gamma(\frac{d}{2(d-1)}))^2\Gamma(\frac{d+1}{2(d-1)})}\ell\big[1 - 2a\lambda_a\big(d^2 - 5d + 6\big)\big]. \tag{101}$$

If we assume that there exists the following first law like of thermodynamics $T_{ent}\Delta S_{EE} = \Delta M$, then

$$T_{ent} = c\ell^{-1}, \quad \text{with } c = \frac{2(d^2-1)(\Gamma(\frac{d}{2(d-1)}))^2\Gamma(\frac{d+1}{2(d-1)})}{\sqrt{\pi}(\Gamma(\frac{1}{2(d-1)}))^2\Gamma(\frac{1}{(d-1)})}\big[1 + 2a\lambda_a\big(d^2 - 5d + 6\big)\big]. \tag{102}$$

The constant a can be fixed by comparing Eq. (36) with Eq. (C.24) of [18] and there follows $a = \frac{2}{(d-2)(d-3)}$ for $\lambda_a = \lambda_{there}$. In which case

$$c = \frac{2(d^2-1)(\Gamma(\frac{d}{2(d-1)}))^2\Gamma(\frac{d+1}{2(d-1)})}{\sqrt{\pi}(\Gamma(\frac{1}{2(d-1)}))^2\Gamma(\frac{1}{(d-1)})}[1 + 4\lambda_a]. \tag{103}$$

At least for $d = 4$, we find that the quantity c is positive, which is expected. Recall, that the minimum value of $\lambda_a = -7/36$, follows from [29–31]. In fact, for any d with positive T_{ent} requires us to set[13] $\lambda_a \geqslant -1/4$.

By doing the fluctuation to other parts of the metric component, the authors of [35] found a modified first law like relation which involve both the 'entanglement temperature' and 'entanglement pressure'. For non-conformal theories the T_{ent} is obtained in [37] and [38].

5.2. *Fluctuation for non-zero λ_1 and Λ*

Let us calculate the change in the entanglement entropy and the mass due to the fluctuation in the geometry for $d = 6$. Doing the one parameter fluctuation, m, as done before in Eq. (93), we find that the change in entropy comes out as

$$\begin{aligned}\Delta S_{EE} = {} & 9L^4R_0^5 m\ell^2\left(187 - \frac{6732\lambda_1}{R_0^2} + \frac{8040\Lambda}{R_0^4}\right)\left(\Gamma\left(\frac{11}{10}\right)\right)^2 \\ & \times\left(\frac{\Gamma(\frac{1}{10})\Gamma(\frac{6}{5})\Gamma(\frac{8}{5})\Gamma(\frac{17}{10}) - \Gamma(\frac{3}{5})\Gamma(\frac{7}{10})\Gamma(\frac{11}{10})\Gamma(\frac{11}{5})}{1496G_N\sqrt{\pi}\Gamma(\frac{1}{10})\Gamma(\frac{7}{10})(\Gamma(\frac{8}{5}))^3\Gamma(\frac{17}{10})}\right).\end{aligned} \tag{104}$$

To this order, we can again read out the t–t component of the energy–momentum tensor from [32]

$$T_{ab} = \frac{1}{8\pi G_N}\left(K_{ab} - KG_{ab} + 2\lambda(3J_{ab} - JG_{ab}) + 3\tilde{\Lambda}(5P_{ab} - PG_{ab}) + \frac{(d-1)}{\tilde{R}}G_{ab}\right). \tag{105}$$

[13] However, for $\lambda_a \leqslant -1/4$, we need to find an interpretation of T_{ent}.

We shall expand $\tilde{R} = R + \lambda R_1 + \tilde{\Lambda} R_2$ to linear order such that as we take the couplings to zero, we do get back the size of the AdS spacetime, R. The sizes R_1 and R_2 will be determined by demanding that T_{tt} becomes finite as we approach the boundary. Or in the limit of $m \to 0$, the T_{tt} component should vanish as well [33]. It gives $R_1 = 8/R_0$ and $R_2 = -72/(5R_0^3)$. Using all these ingredients into Eq. (96), we find the mass becomes

$$\Delta M = \frac{5L^4 R_0^5 m \ell}{16\pi G_N}\left(1 - \frac{24\lambda}{R_0^2} + \frac{72\tilde{\Lambda}}{R_0^4}\right). \tag{106}$$

Now let us set the following relation between the bulk couplings λ and $\tilde{\Lambda}$ with that appears on the holographic entanglement entropy functional λ_1 and Λ following [18]

$$\frac{\lambda}{b} = \frac{\lambda_1}{a} \equiv \lambda_a R^2, \quad b = \frac{a}{2}, \qquad \frac{\Lambda}{X_1} = \frac{\tilde{\Lambda}}{X_2} \equiv \Lambda_a R^4, \quad X_1 = 3X_2 \tag{107}$$

with a and X_1 are real numbers. The size of the AdS radii are related as

$$R_0 = \frac{R}{\sqrt{f_\infty}}, \quad \text{where } 1 - f_\infty + \lambda_a f_\infty^2 - \Lambda_a f_\infty^3 = 0, \tag{108}$$

Note that while writing down such an equation, we have already used the relation between λ, $\tilde{\Lambda}$ and λ_1, Λ as written above. To linear order in the coupling we take $R_0 = R(1 - \frac{1}{2}\lambda_a + \frac{1}{2}\Lambda_a)$. Finally, taking the ratio

$$\frac{\Delta S_{EE}}{\Delta M} = \left(\frac{187 - 4488a\lambda_a + 3552X_1\Lambda_a}{13\,090}\right)\sqrt{\pi}\,\ell\,\frac{\Gamma(\frac{1}{5})(\Gamma(\frac{1}{10}))^2}{\Gamma(\frac{7}{10})(\Gamma(\frac{3}{5}))^2}. \tag{109}$$

The first law like of thermodynamics follows, $T_{ent}\Delta S_{EE} = \Delta M$, if we identify

$$T_{ent} = c\ell^{-1}, \quad \text{with } c = \frac{70\Gamma(\frac{7}{10})(\Gamma(\frac{3}{5}))^2}{187\sqrt{\pi}\,\Gamma(\frac{1}{5})(\Gamma(\frac{1}{10}))^2}(187 + 4488a\lambda_a - 3552X_1\Lambda_a) \tag{110}$$

Let us fix the constants a and X_1 by comparing Eq. (36) with Eq. (C.24) of [18]. It follows that $a = 1/6$ and $X_1 = -1/8$ for $d = 6$, after identifying the couplings as $\lambda_a = \lambda_{there}$ and $\Lambda_a = \mu_{there}$. In which case

$$c = \frac{70\Gamma(\frac{7}{10})(\Gamma(\frac{3}{5}))^2}{187\sqrt{\pi}\,\Gamma(\frac{1}{5})(\Gamma(\frac{1}{10}))^2}(187 + 748\lambda_a + 444\Lambda_a). \tag{111}$$

Positivity of T_{ent} along with $\lambda_a \geqslant -1/4$ requires us to set $\Lambda \geqslant 0$.

5.3. Two parameters fluctuation

Now, we include the second parameter and study the change in the entanglement entropy as a function of these two parameters, m and q^2, to the AdS geometry. Let us, write down the geometry with fluctuation as follows

$$g_{tt} = \frac{R^2}{r^2}\left(1 - mr^d + q^2 r^{2(d-1)}\right), \qquad g_{xx} = \frac{R^2}{r^2}, \qquad g_{rr} = \frac{R^2}{r^2}\left(1 + mr^d - q^2 r^{2(d-1)}\right),$$
$$\Lambda_c = -\frac{(d-1)(d-2)}{2R^2}. \tag{112}$$

The original motivation to take such a form of the geometry is to compute the entanglement entropy with electric charges for RN-AdS black hole. But, it is difficult, in practice, to carry out the radial integration involved, analytically, in the calculation of the entanglement entropy. Hence, we shall treat m and q^2 as small parameters

$$m\ell^d \ll 1, \qquad q^2\ell^{2(d-1)} \ll 1. \tag{113}$$

With this kind of fluctuation, we shall compute the entanglement entropy. In fact, this computation is very easy to do, in the limit of vanishing of all the λ_i's in Eq. (63). The radial integral will be performed from the UV cutoff, ϵ, to the turning point, $r_\star$. Moreover, the size ℓ is related to the turning point, $r_\star$. We obtain the entanglement entropy in terms of ℓ as

$$\begin{aligned} S_{EE} = {} & \frac{L^{d-2}R^{d-1}}{2G_N(d-2)\epsilon^{d-2}} - L^{d-2}R^{d-1}\frac{2^{d-3}\pi^{\frac{d-1}{2}}}{(d-2)G_N}\ell^{2-d}\left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\right)^{d-1} \\ & + \frac{m(d-1)L^{d-2}R^{d-1}}{32G_N\sqrt{\pi}(d+1)}\ell^2\frac{(\Gamma(\frac{1}{2(d-1)}))^2}{(\Gamma(\frac{d}{2(d-1)}))^2}\frac{\Gamma(\frac{d}{(d-1)})}{\Gamma(\frac{d+1}{2(d-1)})} \\ & + q^2L^{d-2}R^{d-1}\ell^d\frac{\Gamma(\frac{d}{2(d-1)})}{8G_N\Gamma(\frac{1}{2(d-1)})}\left(\frac{1}{\sqrt{\pi}}\left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\right)^{d-2} - (4\pi)^{\frac{1-d}{2}}\right). \end{aligned} \tag{114}$$

For $q = 0$, it is easy to see that

$$\frac{\ell^2\partial_\ell^2 S_{EE} - \ell\partial_\ell S_{EE}}{\ell^2} = \ell\partial_\ell\left(\ell^{-1}\partial_\ell S_{EE}\right) < 0, \tag{115}$$

for any $d \geqslant 3$. Whereas $\ell^2\partial_\ell^2 S_{EE} + \ell\partial_\ell S_{EE}$ is not necessarily negative. So, also the quantity $\ell^2\partial_\ell^2 S_{EE} = \ell\frac{\partial \mathcal{S}_3^\Sigma}{\partial\ell}$.

For $q \neq 0$, the following quantity

$$\frac{\ell^2\partial_\ell^2 S_{EE} - \ell\partial_\ell S_{EE}}{\ell^2} = \ell\partial_\ell\left(\ell^{-1}\partial_\ell S_{EE}\right) < 0, \tag{116}$$

but only for $d \geqslant 5$.

6. Conclusion and open question

The entanglement entropy is supposed to provide us the amount of classical/quantum information stored in a given region. The beautiful idea of [2] has led us a new way to quantify it, using the celebrated AdS/CFT correspondence. In this paper, we used the Jacobson–Myers functional [16] along with the prescription of [2] to compute the entanglement entropy of different kind of systems. Such systems are described by having different amount of symmetries and are called as Lifshitz solutions.

As per [2], one of the important ingredient require to compute the entanglement entropy is the hypersurface whose boundary coincides with the boundary of the given region under study. The explicit form of the hypersurface is found using the prescription of [9] and because of its covariant nature, the hypersurface is independent of the nature of the entangling region but depends on the bulk couplings. The form of the hypersurface is obtained, essentially, by extremizing the Jacobson–Myers functional. Apparently, it is not clear whether this form of the hypersurface holds good even for time dependent geometries as well.

Upon computing the value of the action over the hypersurface which is considered to be of the shape of a strip gives us the desired result of the holographic entanglement entropy, S_{EE}. For a given size, ℓ, the entanglement entropy obeys the following differential equation for AdS spacetime in $d \geqslant 3$

$$\ell^2 \frac{\partial^2 S_{EE}}{\partial \ell^2} + (d-1)\ell \frac{\partial S_{EE}}{\partial \ell} = 0. \tag{117}$$

We have checked that even in the presence of the higher derivative terms to the holographic entanglement entropy functional, the entanglement entropy obeys the above mentioned, simple looking, differential equation for the AdS spacetime with the following caveat. Since, the analytic computation of S_{EE} to the higher orders in the couplings are very cumbersome. So, we have computed S_{EE} only to linear order in the couplings and checked the above mentioned differential equation.

In [21], a useful quantity, "renormalized entanglement entropy" $\mathcal{S}^{\Sigma}_d$ is introduced, with which the authors have suggested to study the rate of flow for a sphere type entangling region. In our case, we have consider the scale R in [21] as ℓ and the surface, Σ, as the strip and define to study the flow as $\ell \frac{\partial \mathcal{S}^{\Sigma}_d}{\partial \ell} < 0$. We have checked with the help of the differential equation obeyed by S_{EE} and the exact form of S_{EE} that such a quantity, $\ell \frac{\partial \mathcal{S}^{\Sigma}_3}{\partial \ell} < 0$, holds true only for $d = 3$.

By going through an example of one parameter fluctuation, m, to the geometry, we have computed the entanglement entropy. From the result, it is highly suggestive to consider $\ell \partial_\ell (\ell^{-1} \partial_\ell S_{EE})$ as the quantity that should give the rate of flow. As the quantity $\ell \frac{\partial \mathcal{S}^{\Sigma}_3}{\partial \ell}$ is not necessarily negative. However, by studying two parameter fluctuations, m and q, we find that such a quantity becomes negative only for $d \geqslant 5$.

It is *a priori* not completely clear whether this gives the (complete) RG flow. Presumably, it is interesting to include the quantum corrections to the entanglement entropy of RT along the lines of [39,40], and find the full RG flow structure, which we leave for future research.

We have, also, studied the first law like of thermodynamics for low excited states with higher derivative term. In which case, the entanglement temperature T_{ent} goes inversely with the size, ℓ. In fact, the proportionality constant is a function of the dimension, d, and the couplings.

Acknowledgement

I am happy to thanks people who have helped me over the years. It is too difficult to write all the names. Thanks to the computer center, IoP, Bhubaneswar for the help.

Appendix A. General study of EE

We shall show the finite term in the entanglement entropy of $d+1$ dimensional hyperscale violating bulk spacetime goes as $1/\ell^{d-2-\gamma(d-1)}$, where ℓ is the size of the strip. This we demonstrate by computing the entanglement entropy of the Lifshitz spacetime when the entangling region is of the strip type. By doing a change of the coordinate, we re-express the Lifshitz geometry as a spacetime where the time coordinate scales linearly then we compute the entanglement entropy of this geometry. As expected, the entanglement entropy computed for these two cases gives the same answer.

To begin with, let us write down the geometry of the Lifshitz spacetime in $d+1$ dimensional spacetime with dynamical exponent z as

$$ds^2_{I,d+1} = R^2\left[-\frac{dt^2}{r^{2z}} + \frac{dx^i\,dx^j\,\delta_{ij}}{r^2} + \frac{dr^2}{r^2}\right], \tag{A.1}$$

where the boundary is at $r=0$ and R is the size of the bulk Lifshitz spacetime. It is easy to see that it has the following scaling symmetry

$$r \to \lambda r, \qquad t \to \lambda^z t, \qquad x^i \to \lambda x^i. \tag{A.2}$$

We can re-write the geometry by doing the following change of coordinates $r^z = \rho$, $t = \tilde{t}/z$, $x^i = \tilde{x}^i/z$, $R/z = \tilde{R}$ as

$$ds^2_{II,d+1} = \tilde{R}^2\left[-\frac{d\tilde{t}^2}{\rho^2} + \frac{d\tilde{x}^i d\tilde{x}^j \delta_{ij}}{\rho^{2/z}} + \frac{d\rho^2}{\rho^2}\right], \tag{A.3}$$

which has the following scaling symmetry

$$\rho \to \lambda^z \rho, \qquad \tilde{t} \to \lambda^z \tilde{t}, \qquad \tilde{x}^i \to \lambda \tilde{x}^i. \tag{A.4}$$

The rationale behind taking such a non-linear scaling of the radial coordinate is that, we do not want to change the fact that we started out with a spacetime which has a dynamical exponent z.

Let us compute the entanglement entropy of these two spacetimes using the proposal of RT. According to the proposal the entanglement entropy is computed for a fixed time for which the $d-1$ dimensional spacelike hypersurface, γ, extremizes the area of this hypersurface. Finally, the entanglement entropy is conjectured to take the following form, $S_\gamma = \frac{Area(\gamma)}{4G_N}$.

Let us assume that the precise form of the hypersurface is determined by the function $r(x_1)$ and $\rho(x_1)$, in which case, the induced metric for these two cases are

$$\begin{aligned} ds^2_{d-1}(\gamma_I) &= R^2\left[\left(\left(\frac{dr}{dx_1}\right)^2 + 1\right)\frac{dx_1^2}{r^2} + \frac{dx_2^2 + \cdots + dx_{d-1}^2}{r^2}\right], \\ ds^2_{d-1}(\gamma_{II}) &= \tilde{R}^2\left[\left(\left(\frac{d\rho}{d\tilde{x}_1}\right)^2 \rho^{\frac{2-2z}{z}} + 1\right)\frac{\tilde{x}_1^2}{\rho^{2/z}} + \frac{d\tilde{x}_2^2 + \cdots + d\tilde{x}_{d-1}^2}{\rho^{2/z}}\right]. \end{aligned} \tag{A.5}$$

Let us consider the entangling region is of the strip type. It means $0 \leqslant x_1 \leqslant \ell$ and $-L/2 \leqslant (x_2, \ldots, x_{d-1}) \leqslant L/2$. In which case, the area becomes

$$\begin{aligned} \mathcal{A}(\gamma_I) &= 2R^{d-1}L^{d-2}\int \frac{dr}{r^{d-1}}\sqrt{1 + (dx_1/dr)^2} \\ \mathcal{A}(\gamma_{II}) &= 2\tilde{R}^{d-1}\tilde{L}^{d-2}\int \frac{d\rho}{\rho^{(d-1)/z}}\sqrt{\rho^{\frac{2-2z}{z}} + (d\tilde{x}_1/d\rho)^2}, \quad \tilde{L} \equiv zL \end{aligned} \tag{A.6}$$

Now we can extremize the area to find the hypersurface in both the cases, and are given as

$$\frac{dx_1}{dr} = \frac{(r/r_\star)^{(d-1)}}{\sqrt{1 - (r/r_\star)^{2(d-1)}}}, \qquad \frac{d\tilde{x}_1}{d\rho} = \frac{\rho^{\frac{d-z}{z}}\rho_\star^{\frac{-(d-1)}{z}}}{\sqrt{1 - (\rho/\rho_\star)^{\frac{2(d-1)}{z}}}}, \tag{A.7}$$

where $r_\star$ and $\rho_\star$ are determined as the place where the velocities diverges, i.e., $(\frac{dx_1}{dr})_{r_\star} \to \infty$ and $(\frac{d\tilde{x}_1}{d\rho})_{\rho_\star} \to \infty$, respectively. Substituting the solution into the area gives

$$\mathcal{A}(\gamma_I)=2R^{d-1}L^{d-2}\int_{\epsilon}^{r_\star}\frac{dr}{r^{d-1}}\frac{1}{\sqrt{1-(r/r_\star)^{2(d-1)}}},$$
$$\mathcal{A}(\gamma_{II})=2\tilde{R}^{d-1}\tilde{L}^{d-2}\int_{\varepsilon}^{\rho_\star}\frac{d\rho}{\rho^{(d+z-2)/z}}\frac{1}{\sqrt{1-(\rho/\rho_\star)^{\frac{2(d-1)}{z}}}},\tag{A.8}$$

where we have put an UV cutoff, ϵ and ε, to regulate the presence of divergences. As expected, for unit dynamical exponent there is no difference between these two areas. Let us use the following result for $r_\star>0$

$$\int\frac{dr}{r^n}\frac{1}{\sqrt{1-(r/r_\star)^{2m}}}=\frac{r^{1-n}}{1-n}\,{}_2F_1\left[\frac{1}{2},\frac{1-n}{2m},1+\frac{1-n}{2m},\left(\frac{r}{r_\star}\right)^{2m}\right],\quad \text{for } n\neq 1,\tag{A.9}$$

where ${}_2F_1[a,b,c,x]$ is the hypergeometric function. It is very interesting to see that in both cases for the area, the factor $\frac{1-n}{m}$ are the same. So, the final form of the area becomes

$$\frac{\mathcal{A}(\gamma_I)}{2}=R^{d-1}L^{d-2}\left(\frac{r^{2-d}}{2-d}{}_2F_1\left[\frac{1}{2},\frac{2-d}{2(d-1)},\frac{d}{2(d-1)},\left(\frac{r}{r_\star}\right)^{2(d-1)}\right]\right)_{\epsilon}^{r_\star}$$
$$=\frac{R^{d-1}L^{d-2}}{(d-2)}\frac{1}{\epsilon^{d-2}}-\frac{R^{d-1}L^{d-2}}{(d-2)}\sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}r_\star^{2-d}$$
$$\frac{\mathcal{A}(\gamma_{II})}{2}=z\tilde{R}^{d-1}\tilde{L}^{d-2}\left(\frac{\rho^{\frac{2-d}{z}}}{2-d}{}_2F_1\left[\frac{1}{2},\frac{2-d}{2(d-1)},\frac{d}{2(d-1)},\left(\frac{\rho}{\rho_\star}\right)^{\frac{2(d-1)}{z}}\right]\right)_{\varepsilon}^{\rho_\star}$$
$$=\frac{\tilde{R}^{d-1}\tilde{L}^{d-2}}{(d-2)}\frac{z}{\varepsilon^{\frac{d-2}{z}}}-z\frac{\tilde{R}^{d-1}\tilde{L}^{d-2}}{(d-2)}\sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\rho_\star^{\frac{2-d}{z}},\tag{A.10}$$

where we have used

$${}_2F_1[a,b,c,1]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \text{for } c\neq 0,-1,-2,\ldots\tag{A.11}$$

Upon computing ℓ, the size of the strip along x_1, as a function of the turning point, $r_\star$ or $\rho_\star$, we find

$$\frac{\ell}{2}=r_\star\sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})},\qquad \frac{\ell}{2}=\rho_\star^{\frac{1}{z}}\sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}.\tag{A.12}$$

Now, substituting all these into the area

$$\frac{\mathcal{A}(\gamma_I)}{2}=\frac{R^{d-1}L^{d-2}}{(d-2)}\frac{1}{\epsilon^{d-2}}-\frac{2^{d-2}R^{d-1}L^{d-2}}{(d-2)\ell^{d-2}}\pi^{\frac{d-1}{2}}\left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\right)^{d-1},$$
$$\frac{\mathcal{A}(\gamma_{II})}{2}=\frac{R^{d-1}L^{d-2}}{(d-2)}\frac{1}{\varepsilon^{\frac{d-2}{z}}}-z^{d-1}\frac{2^{d-2}\tilde{R}^{d-1}L^{d-2}}{(d-2)\ell^{d-2}}\pi^{\frac{d-1}{2}}\left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\right)^{d-1}.\tag{A.13}$$

Now it looks that the finite piece of the entanglement entropy are different for these two cases. As expected for $z=1$, they do give the same answer. Now the question is how to choose which one gives the correct entanglement entropy for the bulk Lifshitz spacetime? Actually, the finite pieces are not different. Recall, that in the second case, we have redefined the size of the Lifshitz spacetime and if we take care of that then they give precisely the same answer. Hence, there is no ambiguity.

Let us move to the computation of the entanglement entropy for the hyperscaling violating theory. In particular, we are interested in the following bulk geometry

$$ds^2_{I,d+1} = R^2 r^{2\gamma}\left[-\frac{dt^2}{r^{2z}} + \frac{dx^i\, dx^j \delta_{ij}}{r^2} + \frac{dr^2}{r^2}\right]. \tag{A.14}$$

In order to have a boundary at $r = 0$, we must take $\gamma < 1$ with $z > 0$. This spacetime have the following scaling behavior

$$r \to \lambda r, \qquad t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad ds \to \lambda^\gamma\, ds. \tag{A.15}$$

As considered previously, we can have another geometry with the same scaling behavior

$$ds^2_{II,d+1} = \tilde{R}^2 \rho^{2\frac{\gamma}{z}}\left[-\frac{d\tilde{t}^2}{\rho^2} + \frac{d\tilde{x}^i d\tilde{x}^j \delta_{ij}}{\rho^{2/z}} + \frac{d\rho^2}{\rho^2}\right], \tag{A.16}$$

for which $\rho \to \lambda^z \rho$, $\tilde{t} \to \lambda^z \tilde{t}$, $\tilde{x}^i \to \lambda \tilde{x}^i$, $ds \to \lambda^\gamma\, ds$. The induced geometries of the $d-1$ hypersurfaces becomes

$$\begin{aligned} ds^2_{d-1}(\gamma_I) &= R^2 r^{2\gamma}\left[\left(\left(\frac{dr}{dx_1}\right)^2 + 1\right)\frac{dx_1^2}{r^2} + \frac{dx_2^2 + \cdots + dx_{d-1}^2}{r^2}\right], \\ ds^2_{d-1}(\gamma_{II}) &= \tilde{R}^2 \rho^{2\frac{\gamma}{z}}\left[\left(\left(\frac{d\rho}{d\tilde{x}_1}\right)^2 \rho^{\frac{2-2z}{z}} + 1\right)\frac{d\tilde{x}_1^2}{\rho^{2/z}} + \frac{d\tilde{x}_2^2 + \cdots + d\tilde{x}_{d-1}^2}{\rho^{2/z}}\right]. \end{aligned} \tag{A.17}$$

Without giving the details, let us quote the area of the hypersurface

$$\begin{aligned} \frac{\mathcal{A}(\gamma_I)}{2} &= R^{d-1} L^{d-2} \int_{\epsilon}^{r_\star} \frac{dr}{r^{(1-\gamma)(d-1)}} \frac{1}{\sqrt{1-(r/r_\star)^{2(1-\gamma)(d-1)}}}, \\ \frac{\mathcal{A}(\gamma_{II})}{2} &= \tilde{R}^{d-1} \tilde{L}^{d-2} \int_{\varepsilon}^{\rho_\star} \frac{d\rho}{\rho^{\frac{d+z-2-\gamma(d-1)}{z}}} \frac{1}{\sqrt{1-(\rho/\rho_\star)^{\frac{2(1-\gamma)(d-1)}{z}}}}, \end{aligned} \tag{A.18}$$

where the turning point, $r_\star$ and $\rho_\star$, are determined as the point where the velocity diverges, as found previously. The explicit form of the velocities are

$$\frac{dx_1}{dr} = \frac{(r/r_\star)^{(1-\gamma)(d-1)}}{\sqrt{1-(r/r_\star)^{2(1-\gamma)(d-1)}}}, \qquad \frac{d\tilde{x}_1}{d\rho} = \frac{\rho^{\frac{d-z-\gamma(d-1)}{z}} \rho_\star^{\frac{-(1-\gamma)(d-1)}{z}}}{\sqrt{1-(\rho/\rho_\star)^{\frac{2(1-\gamma)(d-1)}{z}}}}. \tag{A.19}$$

Doing the integrals we find

$$\ell/2 = r_\star \sqrt{\pi} \frac{\Gamma(\frac{d-\gamma(d-1)}{2(1-\gamma)(d-1)})}{\Gamma(\frac{1}{2(1-\gamma)(d-1)})}, \qquad \ell/2 = \rho_\star^{\frac{1}{z}} \sqrt{\pi} \frac{\Gamma(\frac{d-\gamma(d-1)}{2(1-\gamma)(d-1)})}{\Gamma(\frac{1}{2(1-\gamma)(d-1)})}, \tag{A.20}$$

for $\gamma \neq \frac{d-2}{d-1}$ and $z \neq 1$. Finally, the area integral becomes

$$\frac{\mathcal{A}(\gamma_I)}{2}=\frac{R^{d-1}L^{d-2}}{(d-2-\gamma(d-1))}\frac{1}{\epsilon^{d-2-\gamma(d-1)}}-\frac{R^{d-1}L^{d-2}}{(d-2-\gamma(d-1))(\ell/2)^{d-2-\gamma(d-1)}}$$
$$\times\pi^{\frac{(d-1)(1-\gamma)}{2}}\left(\frac{\Gamma(\frac{d-\gamma(d-1)}{2(1-\gamma)(d-1)})}{\Gamma(\frac{1}{2(1-\gamma)(d-1)})}\right)^{(1-\gamma)(d-1)},\quad \text{for } \gamma\neq\frac{d-2}{d-1}$$

$$\frac{\mathcal{A}(\gamma_{II})}{2}=\frac{R^{d-1}L^{d-2}}{(d-2-\gamma(d-1))}\frac{1}{\varepsilon^{\frac{d-2-\gamma(d-1)}{z}}}-\frac{R^{d-1}L^{d-2}}{(d-2-\gamma(d-1))(\ell/2)^{d-2-\gamma(d-1)}}$$
$$\times\pi^{\frac{(d-1)(1-\gamma)}{2}}\left(\frac{\Gamma(\frac{d-\gamma(d-1)}{2(1-\gamma)(d-1)})}{\Gamma(\frac{1}{2(1-\gamma)(d-1)})}\right)^{(1-\gamma)(d-1)}\quad \text{for } \gamma\neq\frac{d-2}{d-1},\ z\neq 1. \tag{A.21}$$

Let us look at the $\gamma=\frac{d-2}{d-1}$. This case has been analyzed earlier in [26]. Before doing the integral, let us look at the following integral for $r_\star>0$

$$\int\frac{dr}{r}\frac{1}{\sqrt{1-(r/r_\star)^2}}=\mathrm{Log}\left(\frac{r}{1+\sqrt{1-(r/r_\star)^2}}\right)$$
$$\int\frac{dr}{r^3}\frac{1}{\sqrt{1-(r/r_\star)^2}}=-\frac{1}{2r^2}\sqrt{1-(r/r_\star)^2}+\frac{1}{2r_\star^2}\mathrm{Log}\left(\frac{r}{1+\sqrt{1-(r/r_\star)^2}}\right),$$
$$\int\frac{dr}{r^5}\frac{1}{\sqrt{1-(r/r_\star)^2}}=\frac{3}{8r_\star^4}\mathrm{Log}\left(\frac{r}{1+\sqrt{1-(r/r_\star)^2}}\right)-\sqrt{1-\frac{r^2}{r_\star^2}}\left(\frac{2r_\star^2+3r^2}{8r^4r_\star^2}\right). \tag{A.22}$$

Essentially, we are trying to find the cases, where there appears a log term in the area. For this choice of $\gamma=\frac{d-2}{d-1}$, the area up to a divergent term becomes

$$\mathcal{A}(\gamma_I)=2R^{d-1}L^{d-2}\int_\epsilon^{r_\star}\frac{dr}{r}\frac{1}{\sqrt{1-(r/r_\star)^2}}$$
$$\simeq 2R^{d-1}L^{d-2}\,\mathrm{Log}(2r_\star)+\text{divergent term}, \tag{A.23}$$

which is the result found recently in [26].

Appendix B. The form of $x_1'(r)$ from Eq. (61)

We can re-write Eq. (61) as $A_1(r)x_1'^6(r)+A_2(r)x_1'^4(r)+A_3(r)x_1'^2(r)+A_4(r)=0$, where the A_i's are

$$A_1(r)=16g_{xx}^3(r)\left[g_{xx}^{d-1}(r)-c^2\right],\qquad A_4(r)=-16c^2g_{rr}^3(r)$$
$$A_2(r)=16g_{rr}(r)g_{xx}^2(r)\left[2g_{xx}^{d-1}(r)-3c^2\right]-8\lambda_1(d-2)(d-3)g_{xx}^{d-1}(r)g_{xx}'^2(r),$$
$$A_3(r)=16g_{rr}^2(r)g_{xx}(r)\left[g_{xx}^{d-1}(r)-3c^2\right]-8\lambda_1(d-2)(d-3)g_{rr}(r)g_{xx}^{d-2}(r)g_{xx}'^2(r)$$
$$+\lambda_1^2(d-2)^2(d-3)^2g_{xx}^{d-4}(r)g_{xx}'^4(r) \tag{B.1}$$

and c is a constant of integration. The real solution to x_1' is

$$x_1'^2 = -\frac{A_2}{3A_1} + \frac{2^{1/3}(A_2^2 - 3A_1A_3)}{3A_1(9A_1A_2A_3 - 2A_2^3 - 27A_1^2A_4 + \sqrt{(2A_2^3 - 9A_1A_2A_3 + 27A_1^2A_4^2)^2 - 4(A_2^2 - 3A_1A_3)^3})^{1/3}} + \frac{(9A_1A_2A_3 - 2A_2^3 - 27A_1^2A_4 + \sqrt{(2A_2^3 - 9A_1A_2A_3 + 27A_1^2A_4^2)^2 - 4(A_2^2 - 3A_1A_3)^3})^{1/3}}{3 \times 2^{1/3}A_1}. \tag{B.2}$$

We are not writing down the other two complex solutions and to quadratic order in λ_1, it reads as

$$x_1'(r) = \frac{c\sqrt{g_{rr}}}{\sqrt{g_{xx}^d - c^2 g_{xx}}} + \frac{c(d-2)(d-3)\lambda_1 g_{xx}'^2}{4g_{xx}^2\sqrt{g_{rr}}\sqrt{g_{xx}^d - c^2 g_{xx}}} - \frac{c(d-2)^2(d-3)^2(3c^2 g_{xx} - 2g_{xx}^d)g_{xx}'^4}{32 g_{rr}^{3/2} g_{xx}^{4+d}\sqrt{g_{xx}^d - c^2 g_{xx}}}\lambda_1^2 + \mathcal{O}(\lambda_1^3) \tag{B.3}$$

For a geometry like Eq. (51), the solution to linear order in λ_1 reads as

$$\begin{aligned}\pm x_1(r) = c_1 &+ \frac{r^{d\delta-\gamma(d-1)}}{r_\star^{(d-1)(\delta-\gamma)}}\frac{1}{[d\delta-\gamma(d-1)]} \\ &\times {}_2F_1\left[\frac{1}{2}, \frac{d\delta-\gamma(d-1)}{2(d-1)(\delta-\gamma)}, \frac{\delta(3d-2)-\gamma(d-1)}{2(d-1)(\delta-\gamma)}, \left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right] \\ &+ \lambda_1 \frac{(d-2)(d-3)(\gamma-\delta)^2}{[d\delta-\gamma(d-1)]}\frac{r^{d\delta-\gamma(d-1)}}{r_\star^{\delta(d-1)-\gamma(d+1)}} \\ &\times {}_2F_1\left[\frac{1}{2}, \frac{d\delta-\gamma(d+1)}{2(d-1)(\delta-\gamma)}, \frac{(3d-2)\delta-\gamma(3d-1)}{2(d-1)(\delta-\gamma)}, \left(\frac{r}{r_\star}\right)^{2(d-1)(\delta-\gamma)}\right] \\ &+ \mathcal{O}(\lambda_1^2),\end{aligned} \tag{B.4}$$

where the constant c_1 is determined by imposing the condition $x_1(r = r_\star) = 0$.

Appendix C. The couplings

In this section, we give the detailed relations of the couplings used in the paper. We denote the bulk couplings as $(\lambda, \tilde{\Lambda})$

$$\text{Bulk couplings:} \quad \lambda \equiv b\lambda_a R^2, \qquad \tilde{\Lambda} \equiv X_2 \Lambda_a R^4. \tag{C.1}$$

The couplings that appear in the holographic entangling entropy (HEE) functional are (λ_1, Λ)

$$\text{HEE couplings:} \quad \lambda_1 \equiv a\lambda_a R^2, \qquad \Lambda \equiv X_1 \Lambda_a R^4. \tag{C.2}$$

where R is the size of the AdS spacetime. Now the relation between the bulk couplings and the HEE couplings are [18]

$$\lambda_1 = 2\lambda, \qquad \Lambda = 3\tilde{\Lambda} \quad \Rightarrow \quad b = \frac{a}{2}, \qquad X_1 = 3X_2. \tag{C.3}$$

Also it follows from [18] that $a = \frac{2}{(d-2)(d-3)}$ for any d whereas $X_2 = -1/24$ for $d = 6$.

References

[1] J.M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, arXiv:hep-th/9711200;
S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys. Lett. B 428 (1998) 105, arXiv:hep-th/9802109;
E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, arXiv:hep-th/9802150;
O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Phys. Rep. 323 (2000) 183–386, arXiv:hep-th/9905111.
[2] S. Ryu, T. Takayanagi, Phys. Rev. Lett. 96 (2006) 181602, arXiv:hep-th/0603001;
S. Ryu, T. Takayanagi, J. High Energy Phys. 0608 (2006) 045, arXiv:hep-th/0605073.
[3] D.V. Fursaev, J. High Energy Phys. 0609 (2006) 018, arXiv:hep-th/0606184.
[4] M. Headrick, Phys. Rev. D 82 (2010) 126010, arXiv:1006.0047 [hep-th].
[5] H. Casini, M. Huerta, R.C. Myers, J. High Energy Phys. 1105 (2011) 036, arXiv:1102.0440 [hep-th].
[6] A. Lewkowycz, J. Maldacena, J. High Energy Phys. 1308 (2013) 090, arXiv:1304.4926 [hep-th].
[7] T. Hartman, arXiv:1303.6955 [hep-th].
[8] T. Faulkner, arXiv:1303.7221 [hep-th].
[9] V.E. Hubeny, M. Rangamani, T. Takayanagi, J. High Energy Phys. 0707 (2007) 062, arXiv:0705.0016 [hep-th].
[10] A. Bhattacharyya, A. Kaviraj, A. Sinha, J. High Energy Phys. 1308 (2013) 012, arXiv:1305.6694 [hep-th].
[11] B. Chen, J.-j. Zhang, J. High Energy Phys. 1307 (2013) 185, arXiv:1305.6767 [hep-th].
[12] A. Bhattacharyya, M. Sharma, A. Sinha, arXiv:1308.5748 [hep-th].
[13] W. Fischler, S. Kundu, J. High Energy Phys. 1305 (2013) 098, arXiv:1212.2643 [hep-th].
[14] X. Dong, arXiv:1310.5713 [hep-th].
[15] M. Alishahiha, A.F. Astaneh, M.R.M. Mozaffar, arXiv:1311.4329 [hep-th].
[16] T. Jacobson, R.C. Myers, Phys. Rev. Lett. 70 (1993) 3684, arXiv:hep-th/9305016.
[17] J. de Boer, M. Kulaxizi, A. Parnachev, J. High Energy Phys. 1107 (2011) 109, arXiv:1101.5781 [hep-th].
[18] L.-Y. Hung, R.C. Myers, M. Smolkin, J. High Energy Phys. 1104 (2011) 025, arXiv:1101.5813 [hep-th].
[19] R.C. Myers, A. Singh, J. High Energy Phys. 1204 (2012) 122, arXiv:1202.2068 [hep-th].
[20] H. Casini, Class. Quantum Gravity 21 (2004) 2351, arXiv:hep-th/0312238.
[21] H. Liu, M. Mezei, J. High Energy Phys. 1304 (2013) 162, arXiv:1202.2070 [hep-th].
[22] V.E. Hubeny, J. High Energy Phys. 1207 (2012) 093, arXiv:1203.1044 [hep-th].
[23] J. Bhattacharya, M. Nozaki, T. Takayanagi, T. Ugajin, Phys. Rev. Lett. 110 (2013) 091602, arXiv:1212.1164.
[24] W.-z. Guo, S. He, J. Tao, J. High Energy Phys. 1308 (2013) 050, arXiv:1305.2682 [hep-th].
[25] N. Ogawa, T. Takayanagi, T. Ugajin, J. High Energy Phys. 1201 (2012) 125, arXiv:1111.1023 [hep-th].
[26] L. Huijse, S. Sachdev, B. Swingle, Phys. Rev. B 85 (2012) 035121, arXiv:1112.0573 [cond-mat.str-el].
[27] X. Dong, S. Harrison, S. Kachru, G. Torroba, H. Wang, J. High Energy Phys. 1206 (2012) 041, arXiv:1201.1905 [hep-th].
[28] S.S. Pal, J. High Energy Phys. 1304 (2013) 007, arXiv:1209.3559 [hep-th].
[29] D.M. Hofman, J. Maldacena, J. High Energy Phys. 0805 (2008) 012, arXiv:0803.1467 [hep-th];
D.M. Hofman, Nucl. Phys. B 823 (2009) 174, arXiv:0907.1625 [hep-th].
[30] M. Brigante, H. Liu, R.C. Myers, S. Shenker, S. Yaida, Phys. Rev. Lett. 100 (2008) 191601, arXiv:0802.3318 [hep-th].
[31] A. Buchel, R.C. Myers, J. High Energy Phys. 0908 (2009) 016, arXiv:0906.2922 [hep-th].
[32] M.H. Dehghani, R.B. Mann, Phys. Rev. D 73 (2006) 104003, arXiv:hep-th/0602243.
[33] V. Balasubramanian, P. Kraus, Commun. Math. Phys. 208 (1999) 413, arXiv:hep-th/9902121.
[34] R.-G. Cai, Phys. Rev. D 65 (2002) 084014, arXiv:hep-th/0109133.
[35] D. Allahbakhshi, M. Alishahiha, A. Naseh, J. High Energy Phys. 1308 (2013) 102, arXiv:1305.2728 [hep-th].
[36] N. Ogawa, T. Takayanagi, J. High Energy Phys. 1110 (2011) 147, arXiv:1107.4363 [hep-th].
[37] S. He, D. Li, J.-B. Wu, J. High Energy Phys. 1310 (2013) 142, arXiv:1308.0819 [hep-th].
[38] D.-W. Pang, arXiv:1310.3676 [hep-th].
[39] T. Faulkner, A. Lewkowycz, J. Maldacena, arXiv:1307.2892 [hep-th].
[40] T. Barrella, X. Dong, S.A. Hartnoll, V.L. Martin, J. High Energy Phys. 1309 (2013) 109, arXiv:1306.4682 [hep-th].